Advanced Functions

by Kensukeken

$$\frac{\sqrt[n]{a}}{\sqrt[m]{b}} = \frac{\sqrt[nm]{a^m}}{\sqrt[nm]{b^m}} = \sqrt[nm]{\frac{a^m}{b^n}}, b \neq 0$$

ISBN: 123-456-790

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Date: February 26th, 2024

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Review

0.1 Exponent Laws

Product Law

When multiplying two terms with the same base, add the exponents.

$$a^m \cdot a^n = a^{m+n}$$

Quotient Law

When dividing two terms with the same base, subtract the exponents.

$$\frac{a^m}{a^n} = a^{m-n}$$

Power Law

When raising a power to another power, multiply the exponents.

$$(a^m)^n = a^{mn}$$

Zero Exponent Law

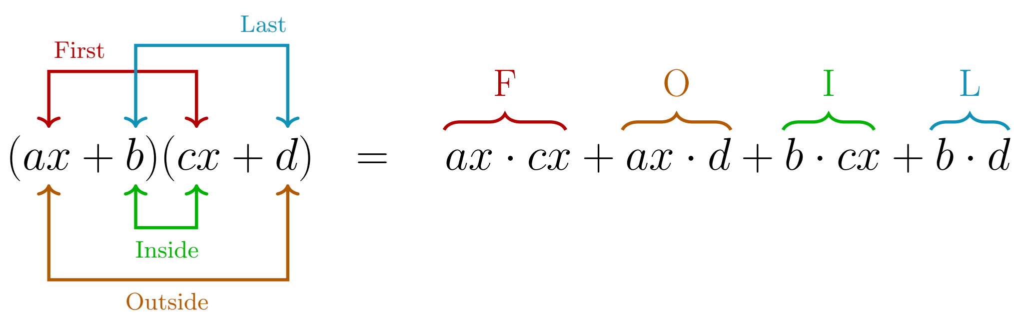
Any nonzero number raised to the power of zero is equal to 1.

$$a^0 = 1$$

Negative Exponent Law

$$a^{-n} = \frac{1}{a^n}$$

FOIL



$$(ax + b)(cx + d) = \underbrace{ax \cdot cx}_{F} + \underbrace{ax \cdot d}_{O} + \underbrace{b \cdot cx}_{I} + \underbrace{b \cdot d}_{L}$$

0.2 Common Number Sets

Natural Numbers ($\mathbb{N}$)

The set of natural numbers is denoted by $\mathbb{N}$ and includes all positive integers from 1 onwards.

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

Integers ($\mathbb{Z}$)

The set of integers is denoted by $\mathbb{Z}$. It includes all whole numbers, both positive and negative, including zero.

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

Rational Numbers ($\mathbb{Q}$)

The set of rational numbers is denoted by $\mathbb{Q}$. It includes all numbers that can be expressed as a fraction of two integers, where the denominator is not zero.

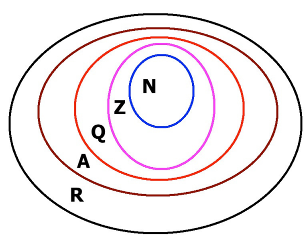
$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$

Real Numbers ($\mathbb{R}$)

The set of real numbers is denoted by $\mathbb{R}$. It includes all rational and irrational numbers, forming the continuum on the number line.

Complex Numbers ($\mathbb{C}$)

The set of complex numbers is denoted by $\mathbb{C}$. It includes all numbers of the form $a + bi$, where a and b are real numbers and i is the imaginary unit.



1 Unit 1

1.1 Power Functions

Linear and Quadratic functions are the two most encountered polynomial functions. Polynomials are defined as follows...
A polynomial expression is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-1}x^{n-2} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0,$$

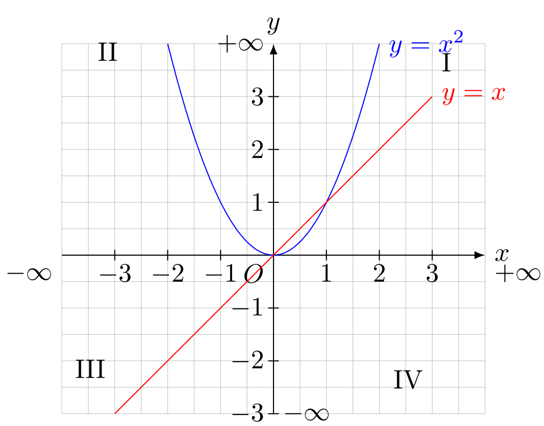
where

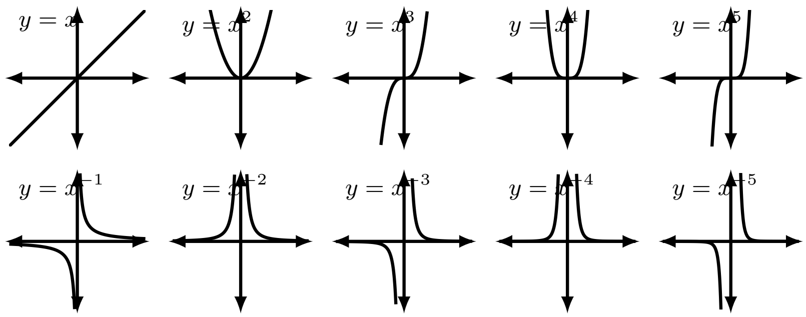
- n is a whole number
- x is a variable
- the coefficients $a_0, a_1, \ldots, a_n$ are real numbers
- the degree of the function is n, the exponent of the greatest power of x
- a_n, the coefficient of the greatest power of x, is the leading coefficient
- a_0, the term without a variable, is the constant term

A polynomial function has the form

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_5x^3 + a_2x^2 + a_1x + a_0$$

Power functions, the simplest polynomial functions, have one term and transform into a general polynomial function when transformed.



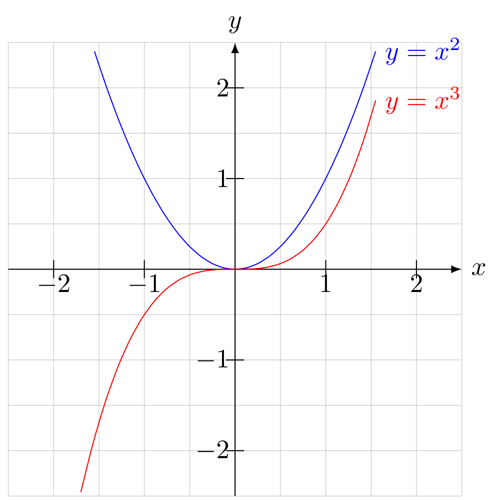


To explore Polynomial functions along with their powers, functions, special names, graphs, domains, ranges, and end behaviors as well as leading terms, refer to the Polynomial.tex file or Power Functions and Polynomial Functions.

Understanding Function Properties

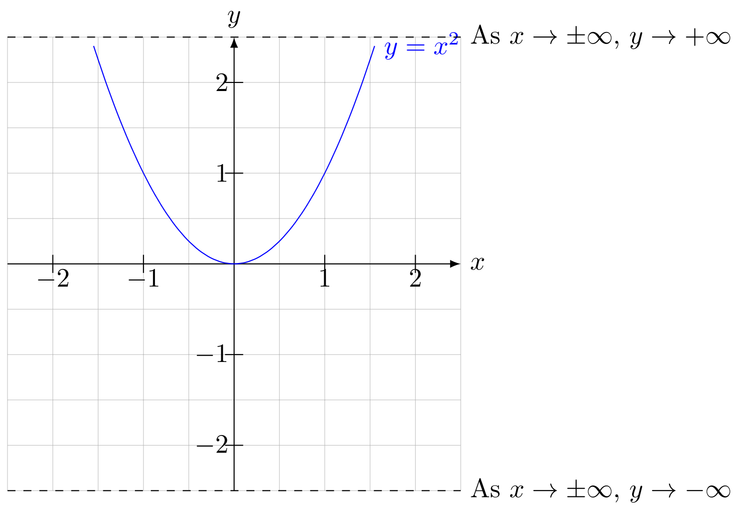
Even and Odd Degree Functions

Even Degree Functions: Graphs that curve from quadrant 3 to quadrant 1. The higher the exponent the closer the curve gets to the y-axis.
Odd Degree Functions: Graphs that make a U-shape. The higher the exponent the U shape gets closer to the y-axis.



Understanding End Behavior

End Behavior: The end behaviour of a function is the behaviour of the y-values as x increases (that is, as x approaches positive infinity, $x \to \infty$) and as x decreases (that is, as x approaches negative infinity, $x \to -\infty$)



Properties of Power Functions

Domain and Range: For power functions, the domain is all real numbers ($x \in \mathbb{R}$), and the range depends on whether the function is even or odd.
Symmetry: Power functions exhibit symmetry properties based on whether they are even or odd functions.

1.1.1 Deriving Polynomial Functions from Data:

x	y	$\Delta f(x)$
1	1	$2-1$
2	2	
3	3	
4	4	$\vdots$
$\vdots$	$\vdots$	$\vdots$
$m-1$	$m-1$	$m-(m-1)$
m	m	$m+1-m$
$m+1$	$m+1$	

The set of first differences for a linear function remains constant.

Union and Intersection

If $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7, \}, whatare A \cup B$ and $A \cap B$?

We have

$$A \cup B = \{1, 2, 3, 5, 7, 9\}$$
$$A \cap B = \{3, 5, 7\}\cdot$$

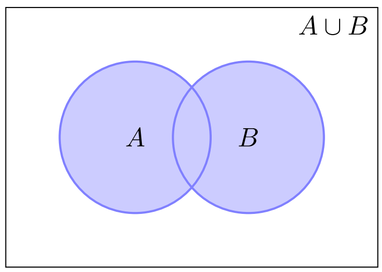


Figure 1: Presentation of sets union by Venn diagram

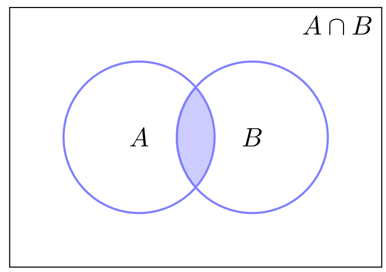
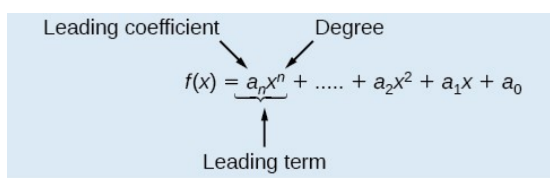


Figure 2: Presentation of sets intersection by Venn diagram

Feel free to learn more in Union and intersections

1.2 Characteristics of Polynomial Function

A general note: Terminology of polynomial functions:



Finite Differences: The nth differences for a polynomial of degree n will all be the same, and this common difference will be equal to the product of n! and the leading coefficient (a_n).

$$\begin{aligned}\text{Common Difference} &= a_n n!(a_n \times n!)\\ &= a_n[n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1]\end{aligned}$$

where,

- a_n represents the nth term of the arithmetic sequence.
- $n!$ represents the factorial of n, which is the product of all positive integers up to n. For example, $5! = 5 \times 4 \times 3 \times 2 \times 1$.

Example:

The finite differences are taken for a polynomial function, and the 6th differences are found to all be -2880. What was the leading coefficient of the polynomial function. Given:

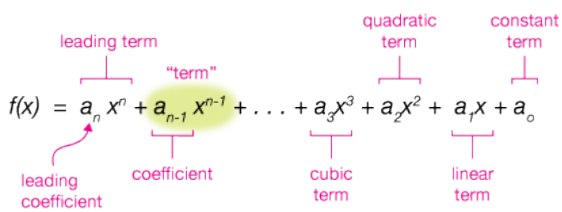
$$\text{Common Difference} = -2880 = a \times 6!$$

We are trying to find the value of a. So, let's solve for a:

$$a = \frac{-2880}{6!} = \frac{-2880}{720} = -4$$

So, the value of a is indeed -4.

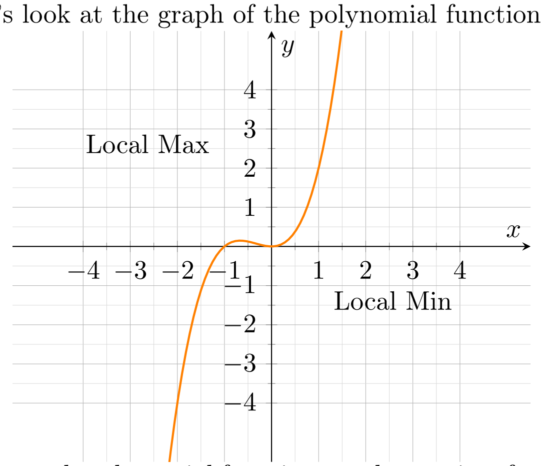
1.2.1 Anatomy of a Polynomial Function:



To learn more about the characteristics of polynomial functions, you can visit the following link: Characteristics of Polynomial Function.

1.2.2 Local Minimum and Maximum Points

Let's look at the graph of the polynomial function defined by $x^3 + x^2$.



Looks more $f(x) = x^3$ than $f(x) = x^2$, but there is a small change we now have "bumps" on the graph.

In general, polynomial function graphs consist of a smooth curve with a series of hills and valleys. The hills and valleys are called turning points. Each **turning point** corresponds to a **local maximum** or **local minimum point**.

Finite Differences: (used to find leading terms and determine degree from a table of values)

Example 1:

Recall for linear functions $f(x) = 3x + 2$ we could make a table of values

x	$f(x)$	First Difference
0	2	
1	5	3
2	8	3
3	11	3
4	14	3
5	17	3

First Difference is constant, so degree is equal to 1 and leading coefficient is 3

Example 2:

x	y	1st Difference	2nd Difference
0	1		
1	6	5	
2	17	11	6
3	34	17	6
4	57	23	6
5	86	29	6

Second Difference is constant, so degree is equal to 2 but the leading coefficient is not 6 it should be 3. So how do we account for this? For a polynomial of degree n, where n is a positive integer, the nth differences:

- are constant (equal)
- have the same sign as the leading coefficient
- are equal to $a(n!)$, where a is the leading coefficient

Factorial (!) means: $n! = n(n-1)(n-2)(n-3)\cdots(2)(1)$
For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.
So for our example above, the second difference is constant, so the degree is equal to 2, but the leading coefficient is $a(n!)$:

$$6 = a(2!) \text{ because } n = 2$$

$$6 = a(2)(1)$$

$$6 = 2a$$

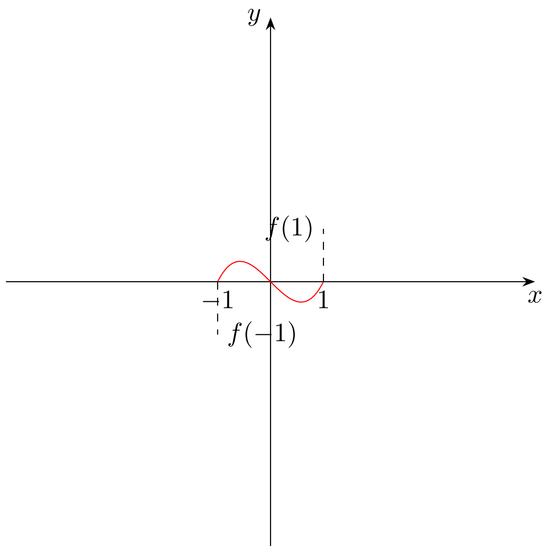
$$3 = a$$

1.2.3 Key Features of Polynomial Functions with Odd Degree

Description

- Odd-degree polynomials have at least one zero, up to a maximum of n x-intercepts, where n is the degree of the function.
- The domain is $x \in \mathbb{R}$ and the range is $y \in \mathbb{R}$.
- They have no absolute maximum point and no absolute minimum point.
- They may have point symmetry.

Graphical Representation



Positive Leading Coefficient

- Graph extends from quadrant 3 to quadrant 1.
- Alternatively, as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$.

Negative Leading Coefficient

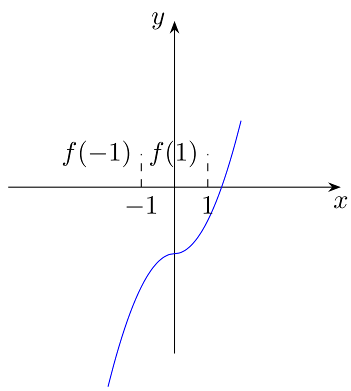
- Graph extends from quadrant 2 to quadrant 4.
- Alternatively, as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$.

1.2.4 Key Features of Polynomial Functions with Even Degree

Description

- Even-degree polynomials may have no zeros, up to a maximum of n x-intercepts, where n is the degree of the function.
- The domain is $\{x \in \mathbb{R}\}$.
- They may have line symmetry.

Graphical Representation



Positive Leading Coefficient

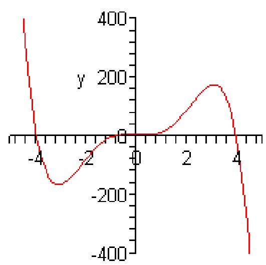
- Graph extends from quadrant 2 to quadrant 1.
- Alternatively, as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to \infty$.
- The range is $\{y \in \mathbb{R} | y \geq a\}$, where a is the absolute minimum value of the function.
- It will have at least one minimum point.
- It will have an absolute minimum point.

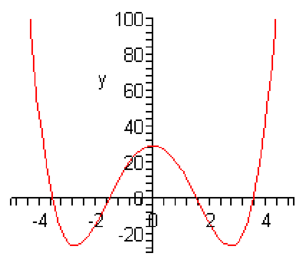
Negative Leading Coefficient

- Graph extends from quadrant 3 to quadrant 4.
- Alternatively, as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to -\infty$.
- The range is $\{y \in \mathbb{R} | y \leq a\}$, where a is the absolute maximum value of the function.
- It will have at least one maximum point.
- It will have an absolute maximum point.

1.3 Graphs of Polynomial Functions

- The function must be in factored form to find the x-intercepts.
- Plot the x-intercepts, and the y-intercept(get h y-intercept by subbing 0 in for x)
- Use an interval test to determine the sign of the polynomial in the intervals divided by the x-intercepts.
- The function $f(x) = (x - 3)(x - 1)(x + 2)^2(x + 5)^3$ is of degree 7. It has 4 intercepts. The zero from the factor $(x + 2)^2$ is repeated and so is said to have order of 2. The zero from the factor $(x + 5)^3$ is repeated and so is said to have order 3.
- The function will pass through the axis at any zero with an odd order, and just skim the x-axis for zeros with an even order.
- For a zero of order 1, the function will pass through the x-axis looking linear.
- For a zero of order 2 the function will pass through the ax-s looking quadratic ... and so on ...





1.4 Symmetry in Polynomial Functions

A polynomial function is called an even function if the exponent of each term of the equation is even. The value of the function would be the same if you subbed in a positive value or its opposite negative value.

$$f(x) = f(-x)$$

Because of this, the function will be symmetric about the y-axis

A polynomial function is called an odd function if the exponent of each term of the equation is odd. The value of the function would have the opposite sign if you subbed in a positive value of its opposite negative value

$$f(-x) = -f(x)$$

Because of this, the function will be symmetric about the origin.

1.5 Transformations of Power Functions

- **Vertical Shift:** Value of C in $f(x) = a[k(x-d)]^n + c$
  - $C > 0$: Shift C units up
  - $C < 0$: Shift C units down
- **Horizontal Shift:** Value of h in $f(x) = a[k(x-d)]^n + c$
  - $d > 0$: Shift $|d|$ units right
  - $d < 0$: Shift $|d|$ units left
- **Vertical Stretch/Compression and Reflection:** Value of a in $f(x) = a[k(x-d)]^n + c$
  - $a > 1$ or $a < -1$: Vertical stretch by a factor of $|a|$
  - $-1 < a < 1$: Vertical compression by a factor of $|a|$
  - $a < 0$: Vertical reflection (reflection in the x-axis)
- **Horizontal Compression/Stretch and Reflection:** Value of k in $f(x) = a[k(x-d)]^n + c$
  - $k > 1$ or $k < -1$: Horizontal compression by a factor of $|k|$
  - $-1 < k < 1$: Horizontal stretch by a factor of $|k|$
  - $k < 0$: Horizontal reflection (reflection in the y-axis)

Note:

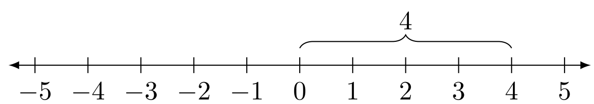
- C and d cause vertical transformations and therefore affect the y-coordinates of the function.
- a and k cause horizontal transformations and therefore affect the x-coordinates of the function.
- When applying transformations to a parent function, make sure to apply the transformations represented by C and d before the transformations represented by a and k.
- We can use the mapping $(x, y) \to \left(\frac{x}{k} + d, ay + c\right)$ to transform every point on the original power function into the new power function

Intro To Absolute Value

Absolute value: $f(x) = |x|$, the definition that "x" is from the origin on the number line.
In general:

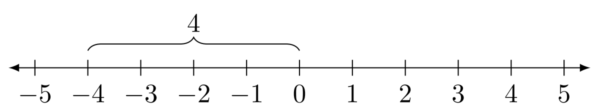
$$|f(x)| = \begin{cases} f(x) & , f(x) \geq 0 \\ -f(x) & , f(x) < 0 \end{cases}$$

The absolute value of a number is its distance from 0. For example, the absolute value of 4 is 4:



This seems kind of obvious. Of course, the distance from 0 to 4 is 4. Where absolute value gets interesting is with negative numbers.

For example, the absolute value of -4 is also 4:



The absolute value symbol

The symbol for absolute value is a bar | on each side of the number.
For example, instead of writing "the absolute value of -6", we can just write |-6|.

Odd functions: A function $f(x)$ is odd if $f(-x) = -f(x)$. Symmetric about the origin or reflection along both the x and y axis.
$f(x) = -2x^3 + x$, is an odd function since $f(-x) = -2(-x)^3 + (-x)$

Note:

$$f(-x) = -f(x)$$

For polynomial functions, the function is an odd function if all exponents are odd.
Even functions: A function $f(x)$ is even if $f(-x) = f(x)$. Symmetric about the y-axis.
i.e $f(x) = 3x^4 + 2x^2 - 2$, is an even function since function if all the exponents are even.

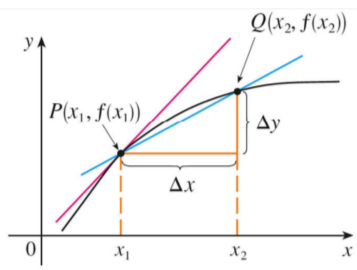
1.6 Slopes of Secants and Average Rate of Change

Key Concepts

- A secant is a straight line that connects two points on a curve.
- Rate of Change (Slope) is a measure of how quickly one quantity (the dependent variable) changes with respect to another quantity (the independent variable). There are two types of rates of change, average and instantaneous.

1.6.1 Average rates of change

represent the rate of change over a specified interval corresponding to the slope of a secant between two points $P_1(x_1, y_1)$ and $P_2(X_2, Y_2)$ on a curve



$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$ This is called a difference quotient

This is the **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$.

An average rate of change can be determined by calculating the slope between two points given in a table of values or by using an equation.

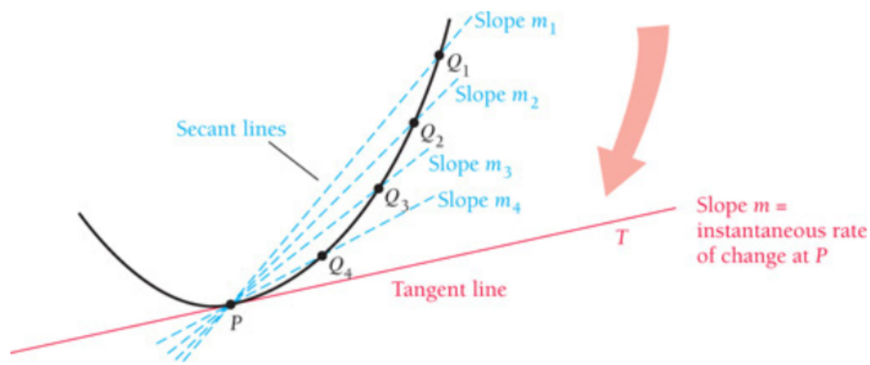
1.7 Slopes of Tangents and Instantaneous Rate of Change

1. A tangent to a curve is a line that intersects a curve at exactly one point.
2. An instantaneous rate of change corresponds to the slope of a tangent at a point on a curve.
3. An approximate value for an instantaneous rate of change at a point may be determined using....

- a graph, either by estimating the slope of a secant passing through that point OR by sketching the tangent and estimating the slope between the tangent point and a second point on the approximate tangent line.
- a table of values, by estimating the slope between the point and a nearby point in the table.
- an equation, by estimating the slope using a very short interval between the tangent point and a second point found using the equation.

1.7.1 Relationship Between the Slope of Secants and the Slope of a Tangent

- As a point Q becomes very close to a tangent point P, the slope of the secant line becomes closer to (approaches) the slope of the tangent line.
- Often an arrow is used to denote the word "approaches". So, the above statement may be written as follows:
- As Q$\rightarrow$ P, the slope of the secant PQ the slope of the tangent at P.
- Thus, the average rate of change between P and Q becomes closer to the value of the instantaneous rate of change at P.



When $x = 0, f(0) = 0$, and when $x = 3, f(3) = 9$. The slope of the secant from $(0, 0)$ to $(3, 9)$ is

$$\begin{aligned}\frac{\Delta f(x)}{\Delta x} &= \frac{9-0}{3-0}\\ &= 3\end{aligned}$$

When $x = 1, f(1) = 1$. The slope from $(1, 1)$ to $(3, 9)$ is

$$\begin{aligned}\frac{\Delta f(x)}{\Delta x} &= \frac{9-1}{3-1}\\ &= 4\end{aligned}$$

When $x = 2, f(2) = 4$. The slope from $(2, 4)$ to $(3, 9)$ is

$$\begin{aligned}\frac{\Delta f(x)}{\Delta x} &= \frac{9-4}{3-2}\\ &= 5\end{aligned}$$

2 Unit 2

2.1 The Remainder Theorem - Part 1

Long Division: Check if you remember how to do long division with constants...

$$
\begin{array}{r|ccccc}
4156\text{r}5 & 6 & 2 & 3 & 4 & 5 \\
15 & 6 & 0 & \downarrow & \downarrow & \downarrow \\
\hline
 & & 2 & 3 & & \\
 & & 1 & 5 & & \\
\hline
 & & & 8 & 4 & \\
 & & & 7 & 5 & \\
\hline
 & & & & 9 & 5 \\
 & & & & 9 & 0 \\
\hline
 & & & & & 5
\end{array}
$$

The process of long division is similar with polynomials.

Example 1: Divide $3x^4 - 12x^3 - 20x^2 - 30x + 2$ **by** $x - 5$

$$3x^4 - 12x^3 - 20x^2 - 30x + 2x - 5$$

You know you are finished when the degree of the remainder is less than the degree of the divisor. For example – we have a linear divisor so our remainder will be constant.

If we state our result in quotient form it would look like this...

$$\frac{3x^4 - 12x^3 - 20x^2 - 30x + 2}{x - 5} = 3x^3 + 3x^2 - 5x - 55 - \frac{273}{(x-5)}$$

In general that would be...

$$\frac{P(x)}{(x-b)} = Q(x) + \frac{R}{(x-b)}$$

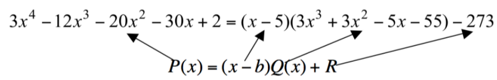
Often I'll want you to conclude a division question with a division statement in the proper form.

Original Polynomial = Divisor x Quotient + Remainder.

In function notation we write:

$$P(x) = (x - b)Q(x) + \mathbb{R}$$

For the example above, the division statement will be



Note: if you expand the right side of the division statement and simplify, you should get what's on the left side.

Example 2: Divide $6z^3 + 13z^2 - 9$ **by** $2z + 3$

You must always place the polynomial in descending powers of the variable. If one power of the variable is missing, it means its coefficient was zero, and you need to put it in as a placeholder.

$$6\text{x}^3 + 13x^2 + 0x - 92x + 3$$

Since the remainder is zero, we know that the divisor went evenly into the polynomial. That makes it a factor of the original polynomial.

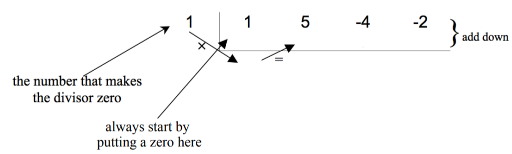
Synthetic Division

Synthetic Division – used when you have a linear divisor To use synthetic division you must have

- a linear divisor where the coefficient of the variable is ONE.
- a polynomial written in descending powers of the variable.
- if you are missing a power of the variable, you must fill in a zero for its coefficient.

Now, we only write the coefficients and fill in the variable when we are finished.

Example 1: Divide $5w^2 - 4w - 2 + w^3$ **by w-1**



The final numbers $(1, 6, 2, 0)$ represent the coefficients of the quotient, starting with the power that is one less than the original polynomial. In this case, our answer will be $w^2 + 6w + 2$, and the last number is the remainder (zero).

So our division statement is: $P(x) = (w - 1)(1w^2 + 6w + 2)$

Example 2: Divide $6w^4 - 3x^3 + 2x - 5$ **by w+3**

$$[x = -3]6x^4 - 3x^3 + 2x - 5$$

2.2 The Remainder Theorem - Part 2

Example 1.

Divide the polynomial $P(x) = x^3 - 2x^2 - 4$ by $x - 2$.
We'll use synthetic division...

$$[x = 2]x^3 - 2x^2 - 4$$

What I want you to notice is that if evaluate $P(2)$ (remember is the value that makes the bracket zero), this is what I get
The final answer is the same as the remainder when we divided. Coincidence? Well, generally if I'm bothering to point out, it is not a coincidence.

The Remainder Theorm

if $P(x)$ is divided by (x-b) and the remainder is constant, then the remainder will be $P(b)$.

Proof: P(x) is a polynomial
$(x - b)$ is the divisor
$Q(x)$ is the quotient
R is the remainder

Then the division statement is

$$P(x) = (x - b)(Q(x)) + R$$

Evaluate $P(b)$

$$\begin{aligned} P(b) &= (b - b)(Q(x)) + R \\ &= \underbrace{O(Q(x))} + R \\ &= R \end{aligned}$$

The General Remainder Theorem

If $P(x)$ is divided by (ax-b) and the remainder is constant, then the remainder will be $P\left(\frac{b}{a}\right)$ where a, b $\in$ I and a $\neq$ 0

Example 2: Find the remainder for each division.

a) $\left(2x^2 - 3x + 7\right) \div (x + 4)$

$$\begin{aligned} P(-4) &= 2(-4)^2 - 3(-4) + 7 \\ &= 51 \end{aligned}$$

$$\boxed{R = 51}$$

b) $\left(4x^3 - 2x^2 + 6x - 1\right) \div (2x - 1)$

$$P\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) - 1$$
$$= 2$$
$$\boxed{R = 2}$$

Example 3.

When the polynomial x3 - 3x2 + kx - 7 is divided by (x-4) the remainder is 29.
What is the value of k?
Always start by filling in what you know...

$$4^3 - 3(4)^2 + k(4) - 7 = 29$$

Solving for k:

$$64 - 48 + 4k - 7 = 29$$
$$4k + 9 = 29$$
$$4k = 20$$
$$k = 5$$

Therefore, the value of k is 5.

Example 4.

For what value of b will the polynomial P(x) = -2x3 + bx2 - 5x + 2
have the same remainder when divided by (x - 2) and (x + 1) If the remainders are the same, we know that P(2) = P(-1) so we sub in and set them equal. Given that $P(2) = P(-1)$, we'll substitute $x = 2$ and $x = -1$ into the polynomial and set the two expressions equal to each other:

For $x = 2$:

$$P(2) = -2(2)^3 + b(2)^2 - 5(2) + 2$$

$$P(2) = -16 + 4b - 10 + 2$$

$$P(2) = 4b - 24$$

For $x = -1$:

$$P(-1) = -2(-1)^3 + b(-1)^2 - 5(-1) + 2$$

$$P(-1) = 2 + b + 5 + 2$$

$$P(-1) = b + 9$$

Setting $P(2)$ equal to $P(-1)$:

$$4b - 24 = b + 9$$

$$4b - b = 9 + 24$$

$$3b = 33$$

$$b = 11$$

Therefore, the value of b is 11.

2.3 The Factor Theorem - Part 1

The Factor Theorem

$(x - b)$ is a factor of $P(x)$ if and only $P(b) = 0$
$(ax - b)$ is a factor of $P(x)$ if and only if $P\left(\frac{b}{a}\right)$

Proof: Given $(x - b)$ is a factor of $P(x)$ then
That means that the quotient $\frac{P(x)}{(x-b)}$ will have a remainder of zero.

Example 1: Is $(x-2)$ a factor of the following polynomials?

a) $x^3 - 7x^2 + 9x + 2$

$$
\begin{aligned}
&= (2)^3 - 7(2)^2 + 9(2) + 2 \\
&= 8 - 28 + 18 + 2 \\
&= 0 \\
&\therefore (x-2) \text{ is a factor}
\end{aligned}
$$

b) $x^3 - 3x^2 + 2x - 5$

$$
\begin{aligned}
&= (2)^3 - 3(2)^2 + 2(2) - 5 \\
&= 8 - 12 + 4 - 5 \\
&= -5 \\
&\therefore (x-2) \text{ is NOT a factor}
\end{aligned}
$$

Integral Zero Theorem

Given the polynomial in factored form:

$$P(x) = (2x-3)(x+4)(x-5)$$

To find the constant term in the expanded polynomial, we multiply all the constant terms of the factors, which are -3, 4, and -5. Thus, the constant term should be:

$$(-3) \times 4 \times (-5) = 60$$

So, if this polynomial were in expanded form, we would know that any zeros would have to be a factor of 60.

In general:

> For $(x-b)$ to be a factor of the polynomial $P(x)$, b must be a factor of the constant term of $P(x)$.

Example 2. Factor the following...

$$x^3 + 2x^2 - 5x - 6$$

We need to divide out a linear factor, then we can employ our methods of factoring quadratics.
By the integral zero theorem, any zeros we find (that are integers) will be factors of 6.
When you check, always start with ± 1.

Feel free to look more explanation on MHF4U U2L3 The Factor Theorem Part 1 at 6:08.

The Rational Zero Theorem

The Rational Zero Theorem provides a useful tool for identifying potential rational roots of a polynomial equation. It states that if a polynomial function

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are integers with $a_n \neq 0$, has any rational roots, then these roots must be of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

In simpler terms, if a polynomial has rational roots, they can be expressed as $\pm\frac{p}{q}$, where p divides evenly into the constant term and q divides evenly into the leading coefficient.

This theorem offers a systematic approach to identifying potential rational roots, which can significantly streamline the process of finding all roots of a polynomial equation.

2.4 Sum and Difference of Cubes

The formulas for factoring sums and differences of cubes are as follows:

Sum of Cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Difference of Cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

These formulas can be derived by expanding the corresponding expressions.

Examples

a) **Sum of Cubes (Numeric Example):** Factor $8a^3 + 27b^3$.

$$8a^3 + 27b^3 = (2a)^3 + (3b)^3$$

$$= (2a + 3b)(4a^2 - 6ab + 9b^2)$$

b) **Difference of Cubes (Numeric Example):** Factor $64x^3 - 27y^3$.

$$64x^3 - 27y^3 = (4x)^3 - (3y)^3$$

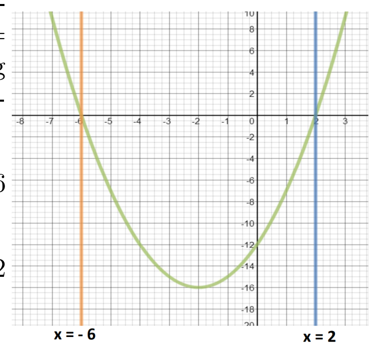
$$= (4x - 3y)(16x^2 + 12xy + 9y^2)$$

These formulas provide a convenient way to factor expressions involving cubes, which can be useful in various algebraic manipulations.

2.5 Solving Polynomial Equations

- A polynomial inequality results when the equal sign in a polynomial equation is replaced with an inequality symbol. $< >$
- The real zeros of a polynomial function, or x-intercepts of the corresponding graph, divide the x-axis into intervals that can be used to solve a polynomial inequality.
- Polynomial inequalities may be solved graphically by determining the x-intercepts and then using the graph to determine the intervals that satisfy the inequality.
- A CAS (computer algebra system) on a graphing calculator may be used to solve a polynomial inequality numerically by determining the roots of the polynomial equation and then testing values in each interval to see if they make the inequality true.

Examine the graph of $f(x) = x^2 + 4x - 12$. The x-intercepts are -6 and 2. These correspond to the zeros of the function $f(x) = x^2 + 4x - 12$. By moving from left to right along the x-axis, we can make the following observations.



- The function is positive when $x < -6$ since the y-values are positive.
- The function is negative when $-6 < x < 2$ since the y-values are negative.
- The function is positive when $x > 2$ since the y-values are positive.

The zeros -6 and 2 divide the x-axis into three intervals: $x < -6, -6 < x < 2$ and $x > 2$. In each interval, the function is either positive or negative. The information can be summarized in a table, as shown below.

Interval	$x < -6$	$-6 < x < 2$	$x > 2$
Sign of Function	$+$	•	$+$

2.6 Solve Factorable Polynomial Inequalities Algebraically

2.6.1 Solving Linear Inequalities

- Solve as you would a regular equation.
- Remember to flip the inequality sign when dividing/multiplying by a negative number.

2.6.2 Solve Polynomial Inequalities

- Factorable inequalities can be solved algebraically by factoring the polynomial, if necessary, and determining the zeros/roots of the function. Then...
  1. Consider all cases, OR
  2. Use intervals and then test values in each interval
- Tables and number lines can help organize intervals to provide a visual clue to solutions.

Example:

Solve the inequality using cases and intervals $(x+3)(2x-3)$ Roots x = - 3, 3/2. The polynomial is already in factored form so we have saved a little work. This will not always be true.

We will start with the pure Algebra based solution.

- Step 1 - determine the number of possible cases for the inequalities We have two brackets being multiplied with the goal being to determine when this function will be greater than zero. i.e when is the function positive The product will be positive when both brackets have a positive value (case 1) or when both brackets have negative values (case 2). So we have two cases
- Step 2 - Solve for both cases (determine when are both true) Case 1

$$\begin{array}{ll} x+3>0 & 2x-3>0 \\ x>-3 & 2x>3 \\ & x>3/2 \end{array}$$

  Both will be positive for numbers less than -3 Case 2

$$\begin{array}{lll} \mathrm{x}+3<0 & 2\mathrm{x}-3<0 & \\ \mathrm{x}<-3 & 2\mathrm{x} & <3 \\ & \mathrm{x} & <3/2 \end{array}$$

  Both will be negative for numbers greater than 3/2
- Step 3 Write your concluding inequality statement

$$\therefore (x+3)(2x-3)>0 \text{ when } \mathrm{x}<-3 \& \mathrm{x}>3/2$$

Let's try this problem a second way using the interval method.

Intervals: Start with the idea that this function has the potential to change from positive to negative values at the roots. We say potential because it could just touch the axis and bend back.

- We will create a table to discuss all regions for the function in space,
- We will test values in these regions in each of the factors to determine the sign of the function.
- We know the roots of this function are at x $= -3, 3/2$ so let us discuss the interval before -3 , between -3 and 3/2, and after 3/2.
- We will still use the logic from the algebra solution, that both factors must either be positive or negative to provide a result that is > 0.

Interval	$x < -3$	$-3 < x < 3/2$	$x > 3/2$
Factors	Try -4	Try 1	Try 4
$(x+3)$	$(-4+3) = -1$ Sign $(-)$	$(1+3) = 4$ Sign$(+)$	$(4+3) = 7$ Sign$(+)$
$(2x-3)$	$[2(-4)-3] = -11$ Sign $(-)$	$[2(1)-3] = -1$ Sign $(-)$	$[2(4)-3] = 5$ Sign $(+)$
Result $(x+3)(2x-3)$	$(+)$	$(-)$	$(+)$

$$\therefore (x+3)(2x-3) > 0 \text{ when x} < -3\&\text{x} > 3/2$$

Example:

The price, P, in dollars, of a stock t years after 2000 can be modelled by the function $P(t) = 0.4t^3 - 4.4t^2 + 11.2t$. When will the price of the stock be more than \$36?

The price, P, in dollars, of a stock t years after 2000 can be modeled by the function:

$$P(t) = 0.4t^3 - 4.4t^2 + 11.2t$$

We want to find the value of t when the price of the stock exceeds \$36. To do this, we set $P(t)$ equal to 36 and solve for t:

$$0.4t^3 - 4.4t^2 + 11.2t - 36 = 0$$

Now, let's find the roots of this equation.

To solve this cubic equation, we can use the **Rational Root Theorem** to identify potential rational roots. The theorem states that if a rational number r is a root of the polynomial equation, then r must be a factor of the constant term (in this case, 36) divided by a factor of the leading coefficient (0.4).

Let's list the possible rational roots:

1. Factors of 36: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$
2. Factors of 0.4: $\pm 0.1, \pm 0.2, \pm 0.4$

Now we'll test each of these roots using synthetic division or long division to find the actual roots. However, I'll spare you the manual calculations and directly provide the solutions:

The roots of the equation are approximately:

$$t_1 \approx 0.5 \quad \text{(rounded to one decimal place)}$$
$$t_2 \approx 4.0 \quad \text{(rounded to one decimal place)}$$
$$t_3 \approx 9.0 \quad \text{(rounded to one decimal place)}$$

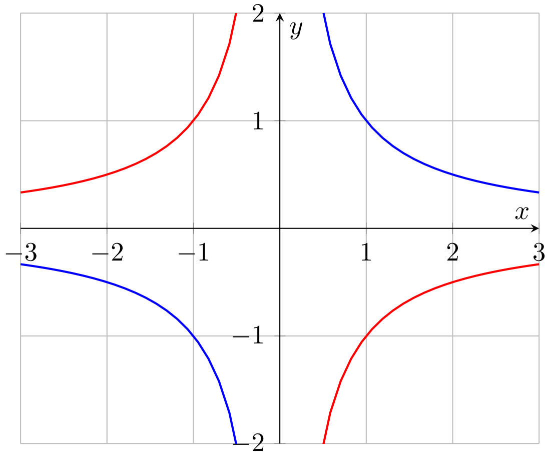
Therefore, the stock price will exceed \$36 at approximately $t = 0.5$, $t = 4.0$, or $t = 9.0$ years after 2000.

3 Unit 3

3.1 Reciprocal of a Linear Function

$$y = mx + b$$

Reciprocal function



x	$\frac{1}{x}$	$-\frac{1}{x}$
-3	-0.33	-0.33
-2	-0.5	-0.5
-1	-1	-1
1	1	-1
2	0.5	-0.5
3	0.33	-0.33

The reciprocal function $f(x) = \frac{1}{x}$ has two hyperbolas in the 1st and 3rd quadrants. As x approaches 0 from either side, $f(x)$ approaches $\pm\infty$.

3.1.1 End Behaviour

$$\begin{aligned} x \to -\infty, y = 0^- \\ x \to +\infty, y \to 0^+ \\ x \to 0^-, y \to -\infty \\ x \to 0^+, y \to +\infty \\ x \in (-\infty, 0) \quad y \downarrow \\ x > 0 \quad y \downarrow \\ x \in (0, +\infty) \quad y \downarrow \end{aligned}$$

3.1.2 Domain and Range

$$D : \{x \in \mathbb{R} | x \neq 0\}, R : \{y \in \mathbb{R} | y \neq 0\}$$

Example 1:

Solve for $f(x) = \frac{1}{-3x+5}$

a) State the domain and range.

b) Describe the behaviour of the function near the vertical asymptote.

c) Describe the end behaviour.

d) Sketch a graph of the function.

Solution

a) **Domain & Range**: The function $f(x) = \frac{1}{-3x+5}$ will be undefined when the denominator equals zero because division by zero is undefined. So, we solve for x:

$$-3x + 5 = 0$$

$$-3x = -5$$

$$x = \frac{5}{3}$$

So, the domain of the function is all real numbers except $x = \frac{5}{3}$. The range is $R : \{y \in \mathbb{R} | y \neq 0\}$

b) **Behaviour near the vertical asymptote**: The vertical asymptote occurs when the denominator approaches zero, which in this case is $x = \frac{5}{3}$. Near this point, the function behaves similarly to $\frac{1}{x}$, approaching positive or negative infinity depending on whether x approaches $\frac{5}{3}$ from the left or the right, respectively.

c) **End behaviour**: As x approaches positive or negative infinity, $f(x)$ approaches zero. This is because the term $-3x$ dominates the function as x becomes very large in magnitude, making the entire fraction approach zero.

d) **Sketch of the graph**: The graph will be a hyperbola with a vertical asymptote at $x = \frac{5}{3}$ and approaching the x-axis as x approaches positive or negative infinity.
$x \to \frac{5}{3}^-, y \to \infty$, $x \in (-\infty, \frac{5}{3})$ increase
$x \to \frac{5}{3}^+, y \to -\infty$, $x \in (\frac{5}{3}, +\infty)$ increase

Example 2:

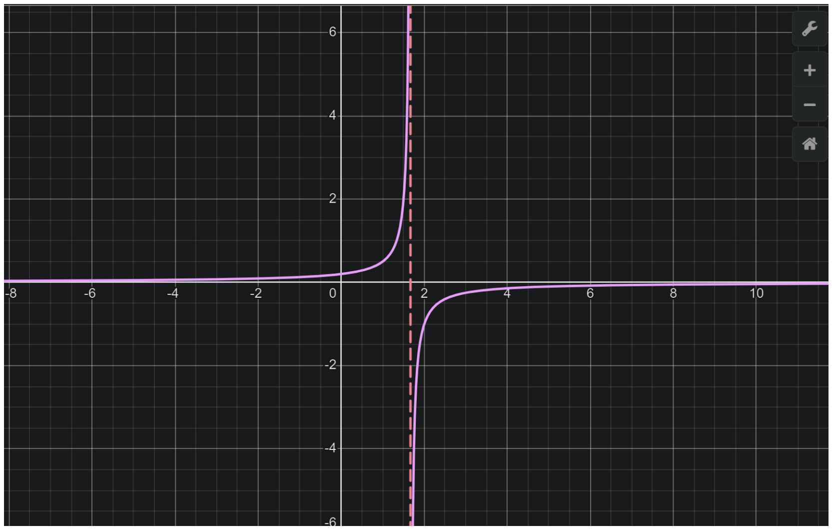
Solve for $f(x) = \frac{1}{(x+1)(x-3)}$

a) State the domain and range.

b) Describe the behaviour of the function near the vertical asymptote.

c) Describe the end behaviour.

d) Sketch a graph of the function.



Solution:

a) Domain and Range: The function $f(x) = \frac{1}{(x+1)(x-3)}$ is undefined at $x = -1$ and $x = 3$. So, the domain of the function is all real numbers except $x = -1$ and $x = 3$. The range is all real numbers except when $f(x)$ equals zero.

b) Behaviour near the vertical asymptotes: Near $x = -1$ and $x = 3$, the function behaves like $\frac{1}{x}$, approaching infinity as x approaches these points.

c) End behaviour As x tends to positive or negative infinity, $f(x)$ approaches zero due to the dominance of $(x + 1)$ and $(x - 3)$ in the denominator.

d) Sketch of the graph:

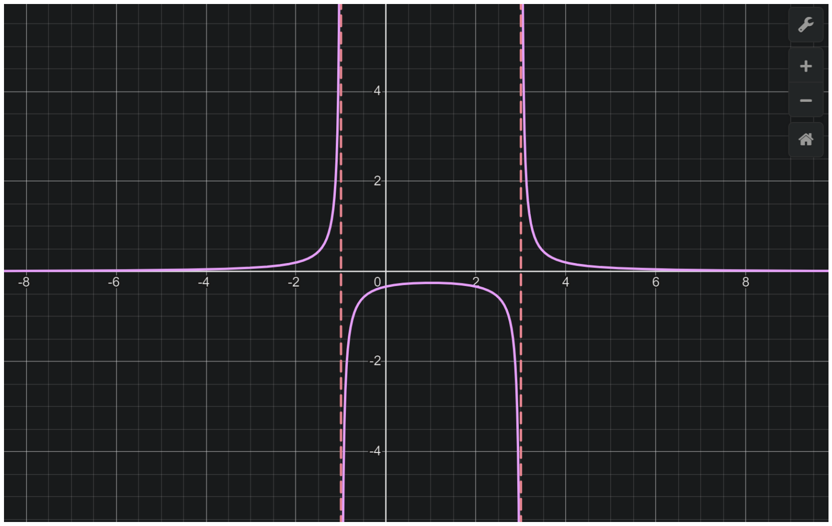
$$\begin{array}{lll} x \to -1^-, y \to +\infty & x \to 3^-, y \to -\infty & x \in (-\infty, -1)(3, +\infty) y > 0 \\ x \to -1^+, y \to +\infty & x \to 3^+, y \to +\infty & x \in (-1, 3) y < 0 \end{array}$$

3.2 Reciprocals of Quadratic Functions

A quadratic function has the form f (x) = ax2 + bx + c in standard form, where a, b and c are real coefficients.
What does the graph of the reciprocal of a quadratic look like?
There are three cases to consider, depending on the factorability of the quadratic.

3.2.1 Asymptotes

Vertical asymptotes occur when the denominator of a rational expression is zero. Thus, the roots of a quadratic expression in the denominator correspond to any vertical asymptotes.



Since a quadratic may have zero, one or two real roots, the reciprocal of a quadratic may have zero, one or two vertical asymptotes.

Like reciprocals of linear functions, horizontal asymptotes can be determined by dividing each term by the highest power, then evaluating as $x \to \infty$.

Example:

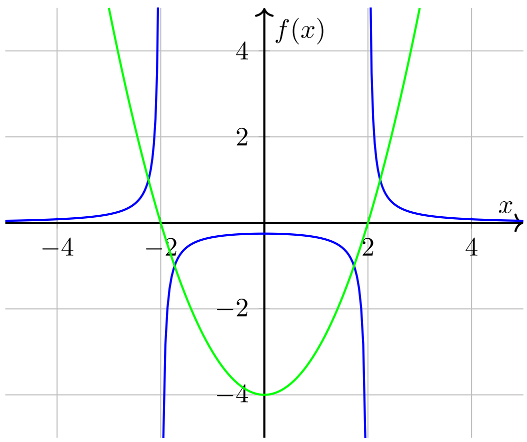
Determine the equations of any asymptotes for

$$f(x) = \frac{1}{x^2 - 4}.$$

After factoring, $f(x) = \frac{1}{(x-2)(x+2)}$. There are two vertical asymptotes: one with equation $x = -2$, and the other $x = 2$. Divide the expression by x^2 and let $x \to \infty$.

$$\begin{aligned} \frac{\frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{4}{x^2}} &= \frac{0}{1 - 0} \\ &= 0 \end{aligned}$$

Asymptotes



$f(x)$ is symmetric about the same axis as $g(x)$. A local maximum occurs on $f(x)$ where there is a local minimum on $g(x)$.

3.2.2 Intercepts

As with any function, the f(x)-intercept can be found by substituting $x = 0$ into its equation.
x-intercepts will occur when the numerator evaluates to zero. If the reciprocal of a quadratic has the form

$$f(x) = \frac{1}{ax^2 + bx + c},$$

then there will always be a horizontal asymptote at f (x) = 0. Verifying the last example, the f (x)-intercept is at $\frac{1}{0-4} = \frac{1}{4}$ and there are no x-intercepts.

3.2.3 Minima/Maxima

Since functions of the form

$$f(x) = \frac{1}{ax^2 + bx + c},$$

have line symmetry, any minimum or maximum point will occur halfway between the two vertical asymptotes.
Substituting in this middle value allows us to determine the coordinate where there is a local min/max. In the previous example, the vertical asymptotes were at $x = -2$ and $x = 2$. Therefore, a local minimum or maximum will occur when

$$x = \frac{-2+2}{2} = 0 \quad \text{or at} \quad \left(0, -\frac{1}{4}\right)$$

3.3 Rational Functions of the Form $f(x)\frac{ax+b}{cx+d}$

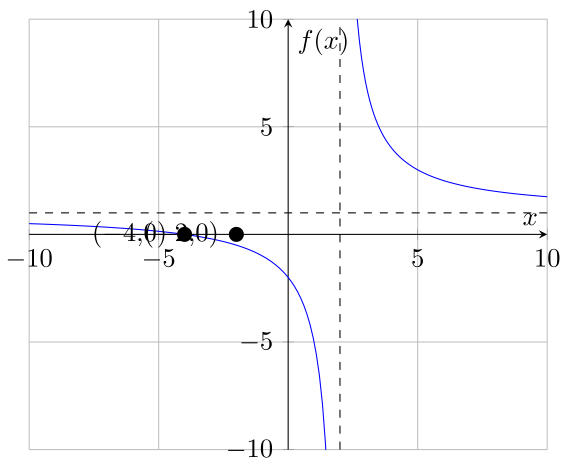
Recall that a rational function is a ratio of two polynomial functions, $p(x)$ and $q(x)$, such that $f(x) = \frac{p(x)}{q(x)}$. Since $q(x) \neq 0$, there will often be some form of discontinuity, such as an asymptote or a hole. In this section, we will investigate rational functions that have the form $f(x) = \frac{ax+b}{cx+d}$. Such functions have predictable properties, making them easy to graph.

Example

Graph the function $f(x) = \frac{x+4}{x-2}$ and describe its properties. There is a vertical asymptote at $x = 2$. Dividing each term by x to find the equation of the horizontal asymptote:

$$\frac{x}{x} + \frac{4}{x} \div \frac{x}{x} - \frac{2}{x} = \frac{1+0}{1-0} = 1$$

A horizontal asymptote occurs at $f(x) = 1$. The x-intercept is at $x = -4$ and the $f(x)$-intercept is at -2.



3.3.1 Putting it Together

To determine the function's behavior to the right of the vertical asymptote, test values of x greater than 2.

3.3.2 Symmetry

$f(4) = 4$ and $f(8) = 2$, resulting in a symmetric graph about the asymptotes.

3.3.3 Complete Graph

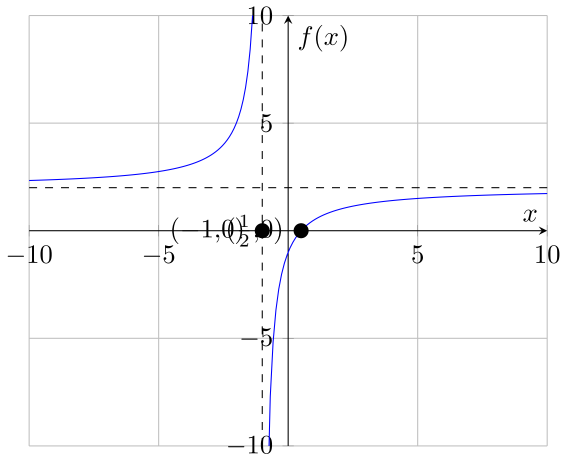
A complete graph of $f(x)$ confirms the symmetry.

Example 2

Graph the function $f(x) = \frac{2x-1}{x+1}$**.** There is a vertical asymptote at $x = -1$. Dividing each term by x to find the equation of the horizontal asymptote:

$$\frac{2x}{x} - \frac{1}{x} \div \frac{x}{x} + \frac{1}{x} = \frac{2-0}{1+0} = 2$$

A horizontal asymptote occurs at $f(x) = 2$. The x-intercept is at $\frac{1}{2}$ and the $f(x)$-intercept is at -1.



3.3.4 Comparison

Comparing the graphs of $f(x) = \frac{x+4}{x-2}$, $g(x) = \frac{x+3}{x-2}$, and $h(x) = \frac{x+2}{x-2}$.

3.3.5 Properties

All three functions have the form $\frac{ax+b}{cx+d}$. As the value of b increases, the function is stretched further from the asymptotes. The value of b has no effect on the vertical and horizontal asymptotes.

Example 3

Determine the equation of a rational function with the following features:

- A vertical asymptote at $x = 3$.
- A horizontal asymptote at $f(x) = 2$.
- An x-intercept at $x = 1$.

Since a vertical asymptote occurs at $x = 3$, let the denominator be $x - 3$. In order for a horizontal asymptote to occur at $f(x) = 2$, and since $c = 1$, the value of a must be 2, since $a/c = 2/1 = 2$. The x-intercept occurs when the numerator is zero, or $2x + b = 0$. Isolating x, this becomes $x = -\frac{b}{2}$. Since the x-intercept is 1, $-\frac{b}{2} = 1$, or $b = -2$. Thus, a possible equation is $f(x) = \frac{2x-2}{x-3}$.

3.4 Rational equations and inequalities

To solve equations involving rational expressions, we have the freedom to clear out fractions before proceeding. After multiplying both sides by the common denominator, we are left with a polynomial equation.

Example:

Solve the equation $\frac{2}{x} + \frac{3x}{x+1} = 4$.

Solution

The common denominator is x(x+1). We multiply both sides by x(x+1) to clear out the fractions.

$$\begin{aligned} \frac{2}{x} + \frac{3x}{x+1} &= 4 \\ x(x+1)\left(\frac{2}{x} + \frac{3x}{x+1}\right) &= x(x+1)(4) \\ x(x+1) \cdot \frac{2}{x} + x(x+1) \cdot \frac{3x}{x+1} &= 4x(x+1) \\ 2(x+1) + 3x(x) &= 4x^2 + 4x \\ 3x^2 + 2x + 2 &= 4x^2 + 4x \\ x^2 + 2x - 2 &= 0. \end{aligned}$$

The quadratic formula gives solutions as $x = \dfrac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3}$

If we look back at the original equation, we notice that there are some numbers that we are not allowed to plug in for x. When $x = 0$ or $x = -1$, the left-hand side of the equation is not defined due to a division by zero issue. Since neither $-1 + \sqrt{3}$ nor $-1 - \sqrt{3}$ have such an issue, they are both solutions.

3.4.1 Inequalities

When faced with nonlinear inequalities, such as those involving general rational functions, we make use of a sign chart. The inequality in the following example is not given in factored form, so we have some work to do.

Example

Solve the inequality $x^2 + 5x \leq -10 - \dfrac{16}{x-2}$

Solution

We'll begin by moving everything to one side, then combining them all together into a single fraction.

$$\begin{aligned} x^2 + 5x &\leq -10 - \frac{16}{x-2} \\ x^2 + 5x + 10 + \frac{16}{x-2} &\leq 0 \\ \left(x^2 + 5x + 10\right) \cdot \left(\frac{x-2}{x-2}\right) + \frac{16}{x-2} &\leq 0 \\ \frac{x^3 + 3x^2 - 20}{x-2} + \frac{16}{x-2} &\leq 0 \\ \frac{x^3 + 3x^2 - 4}{x-2} &\leq 0 \\ \frac{(x-1)(x+2)^2}{x-2} &\leq 0 \end{aligned}$$

Now that the inequality is in a better form for us to work with, we'll build a sign chart like we did in the last example.

x		-2	1	2
$x - 1$	$-$	$-$	$+$	$+$
$(x+2)^2$	$+$	$+$	$+$	$+$
$x - 2$	$-$	$-$	$-$	$+$

We see from the chart that $\dfrac{(x-1)(x+2)^2}{x-2}$ will be negative in $(1, 2)$. At $x = -2$ and $x = 1$ it is zero. The solution is then: $(-2) \cup [1, 2)$.

3.5 Making Connections With Rational Functions and Equation

Special cases occur when a factored rational function has the exact same bracket in both the numerator and denominator. In this case, the bracket would be crossed out on both the top and bottom and a statement reflecting a hole at that point would be made.

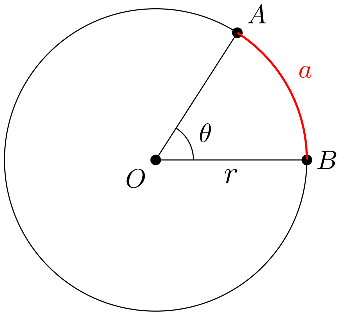
Resources

- Rational equations and inequalities
- Veritcal and horizontal asymptotes

4 Unit 4

4.1 Radian Measure

If you draw a circle with center O and radius r, and you draw two radii OA and OB, then you can define length AB as an arc a and angle AOB as θ.



The Radian Measure is way to express an angle as a ratio. The radian measure of an angle θ is defined as the length of an arc (a) divided by the radius (r).

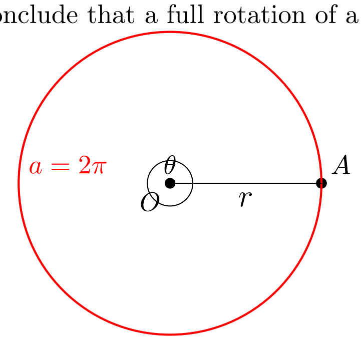
$$\theta = \frac{a}{r}$$

One radius is when the angle of the arc is equal to the radius. What about a full rotation?

Well, we know that the circumstance of a circle is $2\pi r$, so we can substitute this arc length of a full rotation into the definition of a radian.

$$\theta = \frac{2\pi r}{r} = 2\pi rad \approx 6.28rad$$

When using angles in radians, it is convention to drop the units. We can conclude that a full rotation of a circle, then, is 2π



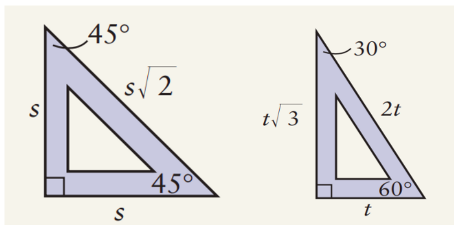
4.1.1 Radian-Degree Conversion

$$\text{Radian} \longrightarrow \text{Degree} \left(\frac{\pi}{180^\circ}\right)$$

$$\text{Radian} \longleftarrow \text{Degree} \left(\frac{180^\circ}{\pi}\right)$$

4.2 Trig Ratios and Special Angles

The triangles found in a geometry set are a $45° - 45° - 90°$ triangle and a $30° - 60° - 90°$ triangle. These triangles can be used to construct similar triangles with the same special relationships among the sides.



$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}, \qquad \csc\left(\frac{2\pi}{3}\right) = \frac{2}{\sqrt{3}},$$
$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}, \qquad \sec\left(\frac{2\pi}{3}\right) = -2,$$
$$\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}, \qquad \cot\left(\frac{2\pi}{3}\right) = -\frac{1}{\sqrt{3}}$$

Table 1: Special Angles and Trigonometric Ratios

Angle	Degrees	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$
0°	0°	0	0	1	0	undefined	1
30°	30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
45°	45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$
60°	60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
90°	90°	$\frac{\pi}{2}$	1	0	undefined	1	undefined

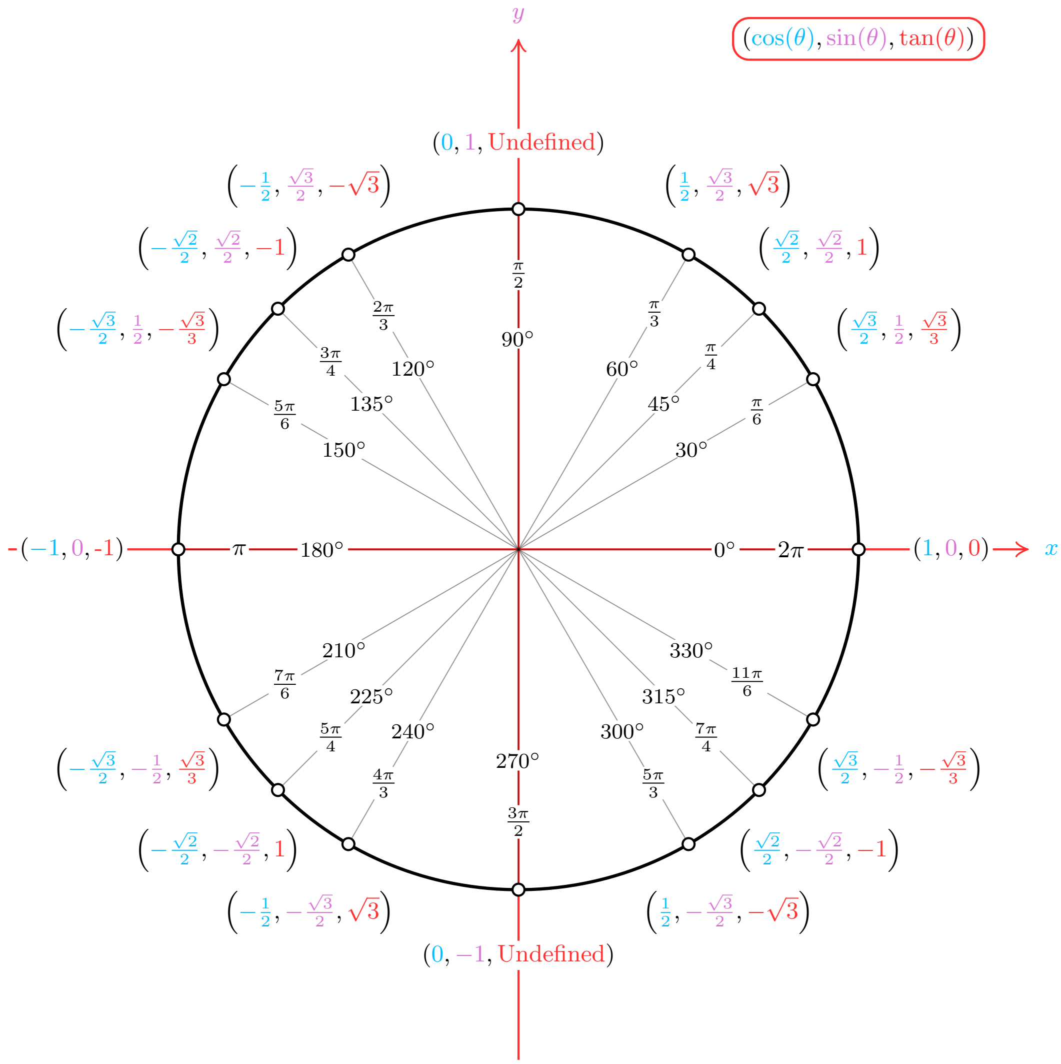
Example

Determine exact values of the six trigonometric ratios for an angle of $\frac{3\pi}{4}$

$$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad \csc\left(\frac{3\pi}{4}\right) = \sqrt{2},$$
$$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}, \quad \sec\left(\frac{3\pi}{4}\right) = -\sqrt{2},$$
$$\tan\left(\frac{3\pi}{4}\right) = -1, \quad \cot\left(\frac{3\pi}{4}\right) = -1.$$

You can use the unit circle and special triangles to determine exact values for the trigonometric ratios of the special angles $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{2}$

Unit Circle



$(\cos(\theta), \sin(\theta), \tan(\theta))$
y
x
(0, 1, Undefined)
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \sqrt{3}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{3}\right)$
(1, 0, 0)
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{3}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -1\right)$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\sqrt{3}\right)$
(0, −1, Undefined)
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, \sqrt{3}\right)$
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1\right)$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{3}\right)$
(−1, 0, -1)
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\frac{\sqrt{3}}{3}\right)$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1\right)$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\sqrt{3}\right)$
$\frac{\pi}{2}$ 90°
$\frac{\pi}{3}$ 60°
$\frac{\pi}{4}$ 45°
$\frac{\pi}{6}$ 30°
0° 2π
$\frac{2\pi}{3}$ 120°
$\frac{3\pi}{4}$ 135°
$\frac{5\pi}{6}$ 150°
π 180°
210° $\frac{7\pi}{6}$
225° $\frac{5\pi}{4}$
240° $\frac{4\pi}{3}$
270° $\frac{3\pi}{2}$
300° $\frac{5\pi}{3}$
315° $\frac{7\pi}{4}$
330° $\frac{11\pi}{6}$

4.3 Equivalent Trigonometric Expression

Definition: Equivalent Trigonometric Expressions

Equivalent trigonometric expressions refer to different algebraic representations of the same trigonometric function or identity. Two trigonometric expressions are considered equivalent if they produce the same value for all possible inputs (angles).

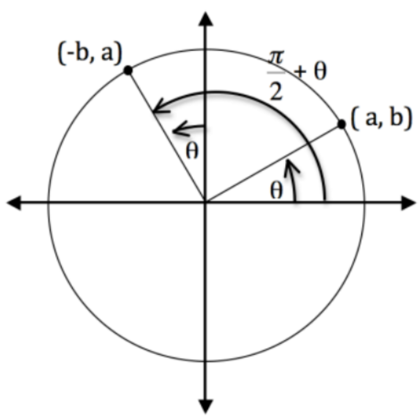
For example, consider the Pythagorean identity:

$$\sin^2(x) + \cos^2(x) = 1$$

This equation demonstrates that the sum of the squares of the sine and cosine of an angle x is always equal to 1. Thus, the expressions $\sin^2(x) + \cos^2(x)$ and 1 are equivalent.

Similarly, expressions such as $\tan(x)$ and $\frac{\sin(x)}{\cos(x)}$ are equivalent, as they represent the same trigonometric function, tangent.

Understanding equivalent trigonometric expressions is crucial for simplifying complex trigonometric equations, identities, and functions, as well as for solving trigonometric equations efficiently.



$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$	$\csc\left(\frac{\pi}{2}+\theta\right) = \sec\theta$	$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$	$\csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta$
$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$	$\sec\left(\frac{\pi}{2}+\theta\right) = -\csc\theta$	$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$	$\sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta$
$\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$	$\cot\left(\frac{\pi}{2}+\theta\right) = -\tan\theta$	$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$	$\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$

4.4 Compound Angle Formulas

The compound angle formulas are as follows:

$$\begin{aligned}
\sin(x+y) &= \sin x \cos y + \cos x \sin y \\
\sin(x-y) &= \sin x \cos y - \cos - \cos x \sin y \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y \\
\cos(x-y) &= \cos x \cos y + \sin x \sin y
\end{aligned}$$

4.4.1 Proof:

Point A is on the terminal arm of angle a, and point B is on the terminal arm of angle $2\pi - b$.

Rotate OA and OB counterclockwise about the origin through an angle b. Point A′ is on the terminal arm of angle $a + b$, and point B′ is on the x-axis.

Join A to B and A′ to B′.

The coordinates of the four points are
A$(\cos a, \sin a)$
A′$(\cos(a+b), \sin(a+b))$
B′$(1, 0)$
B$(\cos(2\pi - b), \sin(2\pi - b))$

You can use equivalent trigonometric expressions from Section 4.3 to write the coordinates of point B as
B$(\cos b, -\sin b)$

Since lengths are preserved under a rotation, A′B′ = AB.

Apply the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y^2 - y_1)^2}$ to A′B′ and AB.

$$\sqrt{[\cos(a+b) - 1]^2 + [\sin(a+b) - 0]^2} = \sqrt{(\cos a - \cos b)^2 + [\sin a - (-\sin b)]^2}$$

$$[\cos(a+b) - 1]^2 + [\sin(a+b) - 0]^2 = (\cos a - \cos b)^2 + [\sin a - (-\sin b)]^2$$ Square both sides.

$$\cos^2(a+b) - 2\cos(a+b) + 1 + \sin^2(a+b) = \cos^2 a - 2\cos a \cos b + \cos^2 b + \sin^2 a + 2\sin a \sin b + \sin^2 b$$ Expand the binomials.

$$\sin^2(a+b) + \cos^2(a+b) + 1 - 2\cos(a+b) = \sin^2 a + \cos^2 a + \sin^2 b + \cos^2 b - 2\cos a \cos b + 2\sin a \sin b$$ Rearrange the terms.

$$1 + 1 - 2\cos(a+b) = 1 + 1 - 2\cos a \cos b + 2\sin a \sin b$$ Apply the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$.

$$-2\cos(a+b) = -2\cos a \cos b + 2\sin a \sin b$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$ Divide both sides by -2.

The addition formula for cosine is usually written as $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

Subtraction Formula for Cosine

The subtraction formula for cosine can be derived from the addition formula for cosine.

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x+(-y)) = \cos x \cos(-y) - \sin x \sin(-y)$$ Substitute $-y$ for y.

$$\cos(x-y) = \cos x \cos(2\pi - y) - \sin x \sin(2\pi - y)$$ From the unit circle, angle $-y$ is the same as angle $(2\pi - y)$.

$$\cos(x-y) = \cos x \cos y - \sin x(-\sin y)$$ $\cos(2\pi - y) = \cos y$; $\sin(2\pi - y) = -\sin y$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

4.4.2 Addition Formula for Sine

Recall the cofunction identities $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ and $\cos x = \sin\left(\frac{\pi}{2} - x\right)$. Apply these and the subtraction formula for cosine.

$$\begin{aligned}
\sin(x+y) &= \cos\left[\frac{\pi}{2} - (x+y)\right] \quad \text{Apply a cofunction identity.} \\
&= \cos\left[\left(\frac{\pi}{2} - x\right) - y\right] \quad \text{Regroup the terms in the argument.} \\
&= \cos\left(\frac{\pi}{2} - x\right)\cos y + \sin\left(\frac{\pi}{2} - x\right)\sin y \quad \text{Apply the subtraction formula for cosine.} \\
&= \sin x \cos y + \cos x \sin y \quad \text{Apply cofunction identities.}
\end{aligned}$$

4.4.3 Subtraction Formula for Sine

The subtraction formula for sine can be derived from the addition formula for sine, following the approach used for the subtraction formula for cosine.

$$\begin{aligned}
&\sin(x + (-y)) = \sin x \cos(-y) + \cos x \sin(-y) \quad \text{Substitute } -y \text{ for } y \\
&\sin(x - y) = \sin x \cos y + \cos x(-\sin y) \\
&\sin(x - y) = \sin x \cos y - \cos x \sin y
\end{aligned}$$

Example

The angles α and θ are located in the FIRST quadrant. If $\sin\theta = \frac{2}{3}$ and $\sin\alpha = \frac{1}{2}$, find $\cos(\theta, \alpha)$

Solution

$$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\sin\alpha$$

We are told that these are first quarter so that we know the signs of all the trig ratios using CAST rule. Other than that we can use right angle to find all other trig ratios for the two angles

$$\begin{aligned}
&= \left(\frac{\sqrt{5}}{3}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{15}}{6} + \frac{2}{6} \\
&= \frac{\sqrt{15}+2}{6}
\end{aligned}$$

4.5 Prove Trig Identities

Quotient Identity : $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Pythagorean Identity: $\sin^2\theta + \cos^2\theta = 1$ (from the unit circle)

Reciprocal Identities :

$$\csc\theta = \frac{1}{\sin\theta}; \quad \sin\theta = \frac{1}{\csc\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}; \quad \cos\theta = \frac{1}{\sec\theta}$$
$$\tan\theta = \frac{1}{\cot\theta}; \quad \cot\theta = \frac{1}{\tan\theta}$$

Compound Angle Formulas:

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$
$$\sin(x-y) = \sin x\cos y - \cos x\sin y$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

Reciprocal of Trigonometry 1.5ex

$$\csc(x) = \frac{1}{\sin(x)}, \quad \tan(x) = \frac{\sin(x)}{\cos(x)}$$
$$\sec(x) = \frac{1}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)}$$

4.6 Applying the Basic Identities

When you prove trig identities, you are trying to do anything you can with the known trig identities in order to transform the two sides and make them look like each other.

You may need to work on BOTH sides of the equal sign, but start first with the side that looks the most complicated. Generally you want to take complicated and make it look more simple.

It is important the a for each step along the way, you state what identity or math operation you have used.

4.6.1 Exercises

a) Prove $\frac{1+\tan x}{1+\cot x} - \frac{1-\tan x}{\cot x - 1}$

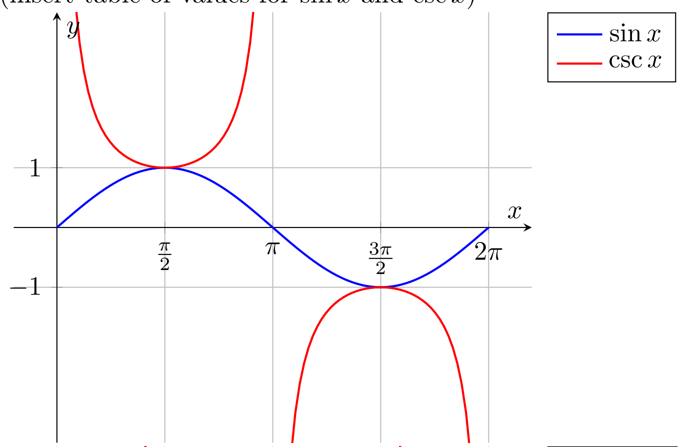
b) Prove $\csc 2x + \cot 2x = \cot x$

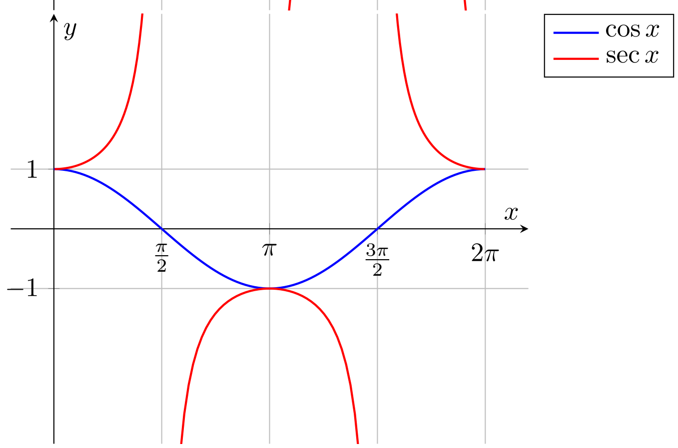
5 Unit 5

5.1 Graphing Trig Functions $\sin x$ & $\csc x$ and $\cos x$ & $\csc x$

In this section, we will be relating the features of $\sin x$ and $\cos x$ to their reciprocals. You will notice that these reciprocals relate to the original function similar to linear reciprocals and quadratic reciprocals.

(insert table of values for $\sin x$ and $\csc x$)





Now, we will look at the key properties of the sine and cosecant functions.

Property	$y = \sin x$	$y = \csc x$
Period	2π	2π
Max Value	1	N/A ∞ is NOT a value
Min Value	-1	N/A ∞ is NOT a value
y-intercept	(0, 0)	N/A
x-intercept	0, π, 2π,... OR $(n\pi, 0)$, $n \in \mathbb{Z}$(1)	N/A
Vertical Asymptote	N/A	$x = n\pi, n \in \mathbb{Z}$ occurs at the x-int of $\sin x$
Amplitude	1	N/A no max or min value
Domain	$\{x \in \mathbb{R}\}$	$\{x \in \mathbb{R} \vert x \neq n\pi, n \in \mathbb{Z}\}$
Range	$\{y \in \mathbb{R} \vert -1 \leq y \leq 1\}$	$\{y \in \mathbb{R} \vert y \leq -1, y \geq 1\}$

Overview: This unit consists of 5 video lessons. Allow no more than 12 class days for this unit, including time for review and to write the test.

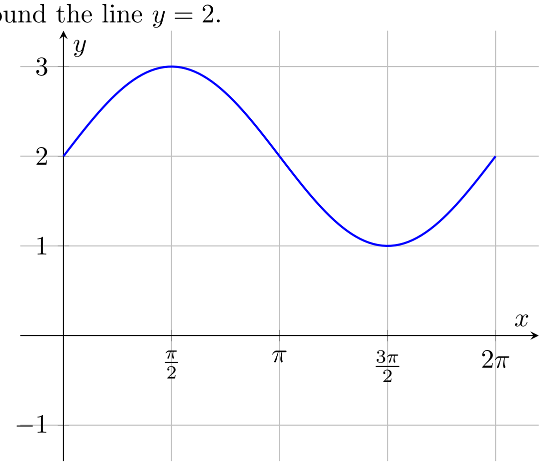
5.2 Graphs of Sine, Cosine and Tangent

$$y = \sin x$$
$$y = \cos x$$
$$y = \tan x$$

Remember that $\sin x$ is defined as the y-coordinate of the unit circle. This will be the distance above or below the x-axis.
For untransformed sinusoidal functions we have the following properties...

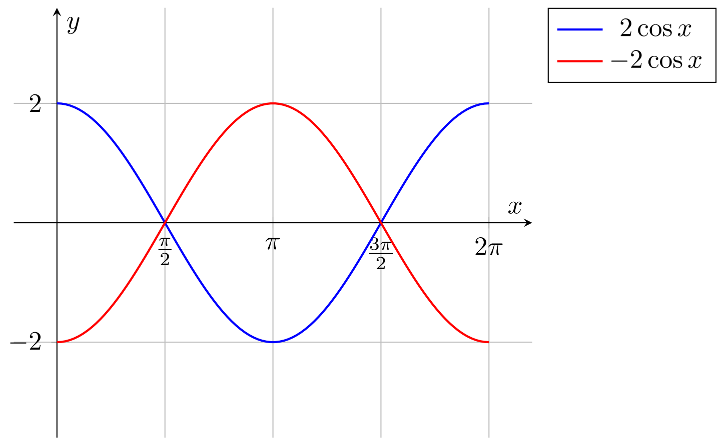
Example 1: Graph $y = \sin x + 2$

Vertical translation: now instead of oscillating around the x-axis, it will oscillate around the line $y = 2$.



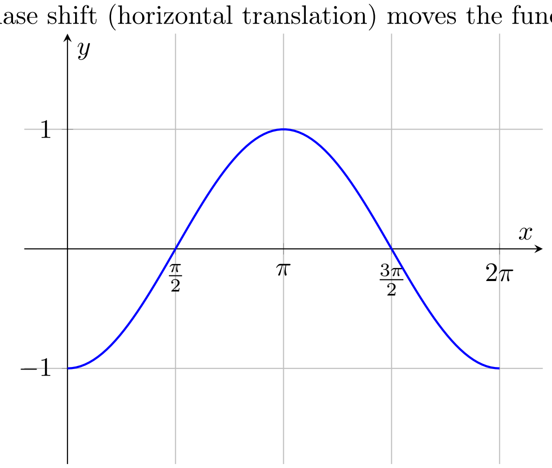
Example 2: Graph $y = 2\cos x$ **and** $y = -2\cos x$

Vertical stretch and reflection.



Example 3: Graph $y = \sin(x - \pi/2)$

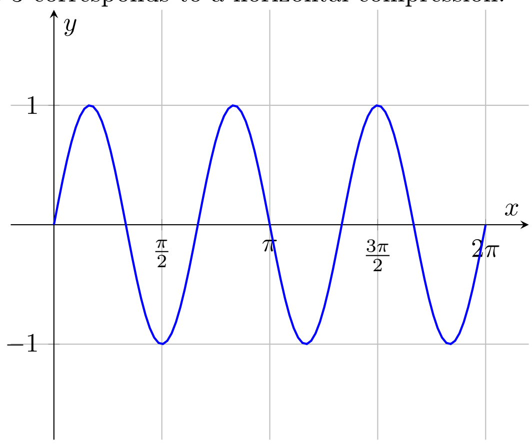
Phase shift (horizontal translation) moves the function left or right.



5.3 Changing the Period

Example 4: Graph the function $y = \sin(3x)$

The 3 corresponds to a horizontal compression.

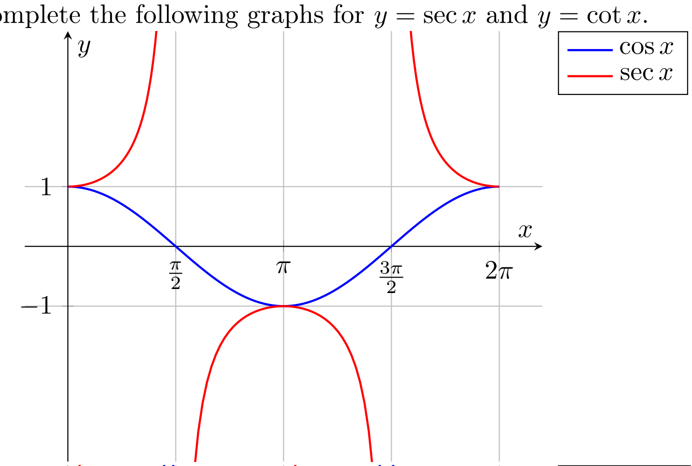


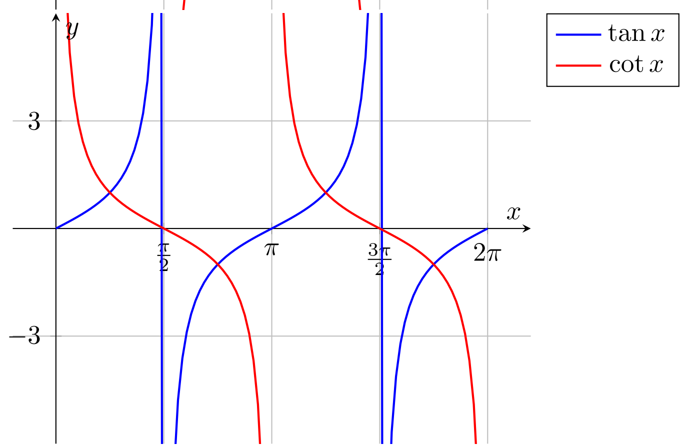
Example 5: Determine the value of k **in** $y = \cos(kx)$ **if the period is** $\pi/2$

$$\text{The period } T = \frac{2\pi}{k} = \frac{\pi}{2} \implies k = 4$$

5.4 Reciprocal Trig Functions

Complete the following graphs for $y = \sec x$ and $y = \cot x$.





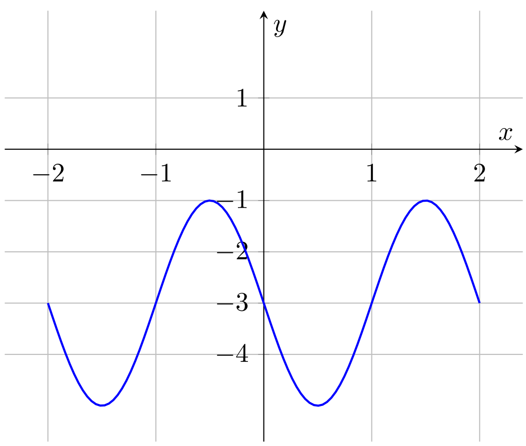
5.5 Sinusoidal Functions

The general form of a sinusoidal function is:

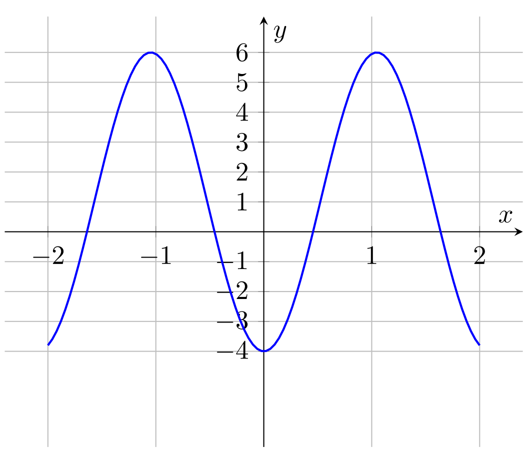
$$f(x) = a\sin[k(x-d)] + c$$

Vertical Stretch: $|a| > 1$
Vertical Compression: $|a| < 1$
Reflection in x-axis: $a < 0$
Horizontal Stretch: $|k| < 1$
Horizontal Compression: $|k| > 1$
Reflection in y-axis: $k < 0$
Phase Shift: horizontal translation
Vertical translation: up $c > 0$, down $c < 0$

Example 1: Graph $y = 2\sin(\pi(x+1)) - 3$



Example 2: Graph $y = -5\cos(3x) + 1$

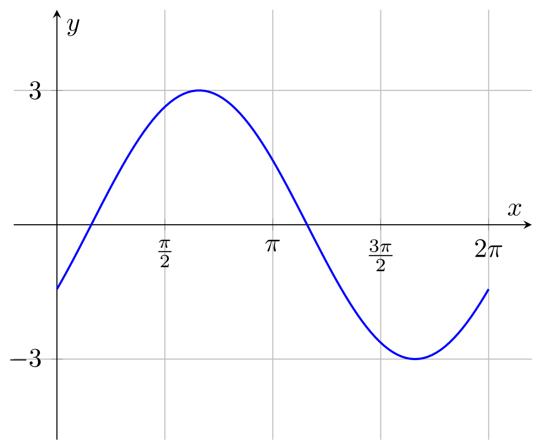


Example 3: Find the form of the cosine curve that has an amplitude of 4, a period of 2π, a left phase shift of $\pi/4$, and a vertical translation of 7.

$$y = 4\cos\left(x + \frac{\pi}{4}\right) + 7$$

Example 4: Sketch the transformation of $f(x) = \sin x$ with an amplitude of 3, period 2π, and a phase shift of 0.5 radians to the right.

$$y = 3\sin(x - 0.5)$$



5.6 Solving Trigonometric Equations

Example 1: Determine the solution to the equation $2\sin x + 1 = 0$ **for** x **in** $[0, 2\pi]$

$$2\sin x + 1 = 0 \implies \sin x = -\frac{1}{2}$$
$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Example 2: Determine the solution to the equation $3(\tan x + 1) = 2$ **for** x **in** $[0, 2\pi]$

$$3(\tan x + 1) = 2 \implies \tan x = -\frac{1}{3}$$
$$x = \arctan\left(-\frac{1}{3}\right), \pi + \arctan\left(-\frac{1}{3}\right)$$

Example 3: Determine the solution to the equation $2\sin^2 x - 3\sin x + 1 = 0$ **for** x **in** $[0, 2\pi]$

$$2\sin^2 x - 3\sin x + 1 = 0 \implies (2\sin x - 1)(\sin x - 1) = 0$$
$$\sin x = \frac{1}{2}, 1$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$$

Example 4: Determine the solution to the equation $2\sec^2 x - 3 + \tan x = 0$ **for** x **in** $[0, 2\pi]$

$$2\sec^2 x - 3 + \tan x = 0 \implies 2(1 + \tan^2 x) - 3 + \tan x = 0 \implies 2\tan^2 x + \tan x - 1 = 0$$
$$(2\tan x - 1)(\tan x + 1) = 0 \implies \tan x = \frac{1}{2}, -1$$
$$x = \arctan\left(\frac{1}{2}\right), \pi + \arctan\left(\frac{1}{2}\right), \arctan(-1), \pi + \arctan(-1)$$

Example 5: Determine the solution to the equation $3\sin x + 3\cos^2 x = 2$ **for** x **in** $[0, 2\pi]$

$$3\sin x + 3\cos^2 x = 2 \implies 3\sin x + 3(1 - \sin^2 x) = 2 \implies 3\sin x + 3 - 3\sin^2 x = 2$$
$$-3\sin^2 x + 3\sin x + 1 = 0 \implies 3\sin^2 x - 3\sin x - 1 = 0$$
$$(3\sin x + 1)(\sin x - 1) = 0 \implies \sin x = -\frac{1}{3}, 1$$
$$x = \arctan(-\frac{1}{3}), \pi + \arctan(-\frac{1}{3}), \frac{\pi}{2}$$

5.7 Making Connections to the "Real World"

Making Connections

We are going to use the graphing calculators for instantaneous rate of change. On the graphing calculator:

- Graph the curve $y = \sin x$ (adjust your window so you see the curve better)
- Use the DRAW button `[2nd PRGM]` to draw on a tangent line
- Press `2nd PRGM`, choose 5: Tangent
- It will take you back to the graph, key in the x-value where you want the tangent line and press enter. Fill in the following table.

Many "real world" applications follow a cyclical or sinusoidal pattern. We can use a sinusoidal curve to model them. Usually we will choose the sine curve and apply a transformation to suit.

Example 1: The number of employees at a City Bicycle Company for each of the last 11 years is shown below. Find a sine curve that will model this data. Use technology to help you.

Year	**Employees**
1	228
2	241
3	259
4	233
5	226
6	209
7	212
8	225
9	240
10	251
11	261

Solution

Amplitude (half the distance from max to min):

$$261 - 209 = 52$$
$$\frac{52}{2} = 26$$

Midline (the horizontal line right between max and min):

$$\frac{261 + 209}{2} = 235$$

Period (for k-value):

$$11 - 3 = 8$$
$$P = \frac{2\pi}{k} \implies 8 = \frac{2\pi}{k}$$
$$8k = 2\pi$$
$$k = \frac{2\pi}{8} \implies k = \frac{\pi}{4}$$

Phase Shift (the x-value when the y is first on the midline):

$$1.3$$

The equation of the sine curve modeling the data is:

$$y = 26 \sin\left(\frac{\pi}{4}(x - 1.3)\right) + 235$$

6 Unit 6

6.1 Logarithms

Introduction to Logarithms

As we saw last class, the logarithm function is the inverse of an exponential. But what exactly does this mean?

If $f(x) = 2^x$ then $f^{-1}(x) = \log_2(x)$

The function returns the value of a power when given an exponent.

The logarithm returns the exponent when given the value of a power.

Example 1: Converting between Logarithmic and Exponential Forms

Write the equivalent log or exponential equation.

$$\begin{aligned} \text{a) } \log_2 16 = 4 &\quad\leftrightarrow\quad 2^4 = 16 \\ \text{b) } 7^2 = 49 &\quad\leftrightarrow\quad \log_7 49 = 2 \\ \text{c) } 5^x = 11 &\quad\leftrightarrow\quad \log_5 11 = x \\ \text{d) } \log_x 42 = 7 &\quad\leftrightarrow\quad x^7 = 42 \end{aligned}$$

Example 2: Evaluating Logarithms

Evaluate the following logs:

$$\begin{aligned} \text{a) } \log_{10} 100 &= 2 \\ \text{b) } \log_2 64 &= 6 \\ \text{c) } \log_2\left(\frac{1}{2}\right) &= -1 \\ \text{d) } \log_3\left(\frac{1}{27}\right) &= -3 \end{aligned}$$

Example 3: Estimating Logarithms

Estimate $\log_3 10$.
The calculator has two log buttons: log and ln (base 10 and base $e \approx 2.7183$, respectively).

Example 4: Using a Calculator to Evaluate Logarithms

Use a calculator to evaluate, then write an equivalent exponential equation.

$$\begin{aligned} &\text{a) } \log 52 \approx 1.716 &&\leftrightarrow && 10^{1.716} = 52 \\ &\text{b) } \log 24 \approx 1.380 &&\leftrightarrow && 10^{1.380} = 24 \\ &\text{c) } \ln 12 \approx 2.485 &&\leftrightarrow && e^{2.485} = 12 \end{aligned}$$

6.2 Power Law of Logarithms

Understanding the Power Law of Logarithms

Evaluate each pair of logarithms on your calculator - What do you notice?

$$\begin{aligned} &\text{a) } 4\log 2 && \log 16 \\ &\text{b) } 2\log 3 && \log 9 \\ &\text{c) } 2\log 5 && \log 25 \end{aligned}$$

Proof: Let $w = \log_b x$. Therefore, $b^w = x$.
What use is the power law?

1. It can help simplify logarithmic expressions.
2. It gives us a method for solving exponential equations.

Example 1: Evaluating Logarithms Using the Power Law

Evaluate the following logarithms:

$$\text{a) } \log_2 8 = 3 \quad (\text{Change } 8 \text{ to } 2^3)$$

$$\text{b) } \log_3 \sqrt{27} = \frac{3}{2}$$

Example 2: Solving Exponential Equations Using Logarithms

Solve by first taking the log of both sides and using the power law of logarithms:

$$\text{a) } 5^t = 15625 \quad \leftrightarrow \quad t\log 5 = \log 15625 \quad \leftrightarrow \quad t = \frac{\log 15625}{\log 5} = 6$$

$$\text{b) } 1000 = 2000(1+0.2)^n \quad \leftrightarrow \quad \frac{1000}{2000} = (1.2)^n \quad \leftrightarrow \quad n = \frac{\log 0.5}{\log 1.2} \approx -3.106$$

Example 3: Evaluating Logarithms Using the Change of Base Formula

Evaluate $\log_3 54$:
First, let the expression equal a variable and turn it into the equivalent exponential expression. This actually gives us the change of base formula...

$$\log_b m = \frac{\log m}{\log b}$$

Example 4: Using the Change of Base Formula

Use the change of base formula to evaluate $\log_8 254$:

$$\log_8 254 = \frac{\log 254}{\log 8} \approx 2.9$$

Example 5: Solving Compound Interest Problems Using Logarithms

An investment of \$2000 earns 2% interest, compounded yearly. A formula to represent this situation is:

$$A = 2000(1.02)^n$$

where A is the amount of the investment and n is the number of years of the investment. How long before the investment doubles?

$$2 = (1.02)^n \quad \leftrightarrow \quad n = \frac{\log 2}{\log 1.02} \approx 35$$

6.3 Equivalent Exponential Expressions

Example 1: Rewriting Exponentials Using a Different Base

Rewrite the following using a base of 3.

$$36 \quad \leftrightarrow \quad 3^6 = (3^2)^3 = 9^3$$

Example 2: Using Logarithms to Find the Value of an Exponent

Use logarithms to find the value of an exponent: $3^x = 11$

$$x = \frac{\log 11}{\log 3} \approx 2.183$$

Example 3: Solve for x by first writing as powers with the same base.

$$27^x = 9^{2x-3} \implies (3^3)^x = (3^2)^{2x-3}$$
$$3^{3x} = 3^{4x-6}$$
$$3x = 4x - 6x = 6$$

6.4 Techniques for Solving Exponential Equations

Techniques for Solving Exponential Equations

Last lesson we solved exponential equations by forcing them to have an equal base and equating their exponents. But what about a situation like this:

$$5^{2x+4} = 3^{x-7}$$

Problem: We can't simply write them with the same base.
Solution: If we take the log of both sides, we can get the variables out of the exponent!

Example 1: Solve for x

$$\begin{aligned} \log 5^{2x+4} &= \log 3^{x-7} \\ (2x+4)\log 5 &= (x-7)\log 3 \\ 2x\log 5 + 4\log 5 &= x\log 3 - 7\log 3 \\ 2x\log 5 - x\log 3 &= -7\log 3 - 4\log 5 \\ x(2\log 5 - \log 3) &= -7\log 3 - 4\log 5 \\ x &= \frac{-7\log 3 - 4\log 5}{2\log 5 - \log 3} \approx -4.54 \end{aligned}$$

Example 2: Solve for x

$$2^x - 2^{-x} = 4$$

There's no immediate method to solve this, we have to make a couple of adjustments in order to solve. Notice it's somewhat similar to a quadratic equation in that the variables are in decreasing order. We are going to multiply through by 2^x and this will become more apparent.

$$2^x(2^x) - 2^x(2^{-x}) = 4(2^x)$$
$$2^{2x} - 1 = 4 \cdot 2^x$$
$$2^{2x} - 4 \cdot 2^x - 1 = 0$$

Let $u = 2^x$:

$$u^2 - 4u - 1 = 0$$
$$u = \frac{4 \pm \sqrt{16+4}}{2} = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5}$$
$$2^x = 2 + \sqrt{5}$$
$$x = \log_2(2 + \sqrt{5}) \approx 1.79$$

Example 3: Using a Half-Life

An archaeological discovery of an unknown plant fossil has 1/8 the amount of radioactive carbon as plants have today. If the half-life of the carbon is 5730 years, how old is the fossil?

$$A = A_0 \left(\frac{1}{2}\right)^{\frac{t}{h}}$$

Given:

$$\frac{1}{8} = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$
$$2^3 = 2^{\frac{t}{5730}}$$
$$3 = \frac{t}{5730}$$
$$t = 3 \times 5730 = 17190 \text{ years}$$

6.5 Log Laws

Log Laws

Remember that there were laws for multiplying and dividing powers with the same base. We need to adapt these for use with logarithms.

$$\log_b(m \cdot n) = \log_b m + \log_b n$$
$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$$

Example 1: Simplify using the laws of logarithms

$$\text{a) } \log 5 + \log 10 = \log(5 \cdot 10) = \log 50$$
$$\text{b) } \log 12 - \log 2 = \log\left(\frac{12}{2}\right) = \log 6$$

Example 2: Simplify each expression

$$\begin{aligned}\text{a) } \log(5a) + \log 10 - \log(2b) &= \log(5a \cdot 10) - \log(2b)\\ &= \log\left(\frac{50a}{2b}\right) = \log\left(\frac{25a}{b}\right)\\ \text{b) } \log x + 6\log y + 3\log z &= \log x + \log(y^6) + \log(z^3)\\ &= \log(x \cdot y^6 \cdot z^3)\end{aligned}$$

Example 3: Evaluate

$$\begin{aligned}\text{a) } \log 50 + \log 10 - \log 5 &= \log(50 \cdot 10) - \log 5\\ &= \log 500 - \log 5 = \log 100 = 2\\ \text{b) } 4\log_{12} 2 + 2\log_{12} 3 &= \log_{12}(2^4) + \log_{12}(3^2)\\ &= \log_{12} 16 + \log_{12} 9 = \log_{12} 144\end{aligned}$$

Example 4: Simplify and state any restrictions

$$\text{a) } \log(2x^2 + 9x - 5) - \log(x+5) = \log\left(\frac{2x^2 + 9x - 5}{x+5}\right)$$

$$\text{b) } \log(x+3) + \log(2x-5) = \log((x+3)(2x-5))$$

6.6 Solving Log Equations

Solving Log Equations

There are three main techniques involved in solving logarithmic equations:

1. Use the definition of a logarithm to rewrite the equation as an exponential. Then solve using the techniques for exponential equations.
2. First simplify using the laws of logarithms, and then rewrite as an exponential to solve.
3. First simplify using the laws of logarithms and then equate the arguments of the logs on both sides of the equal sign.

Example 1: Use the definition to change into an exponential. Solve for n

$$\begin{aligned} \log_3(n^2 - 3n + 5) &= 2 \\ 3^2 &= n^2 - 3n + 5 \\ 9 &= n^2 - 3n + 5 \\ 0 &= n^2 - 3n - 4 \\ (n-4)(n+1) &= 0 \\ n = 4 \text{ or } n &= -1 \end{aligned}$$

Example 2: Simplify and then apply the definition. Solve for p

$$\begin{aligned}
\log{(p+5)} - \log{(p+1)} = 3 \implies \log\left(\frac{p+5}{p+1}\right) &= 3 \\
10^3 &= \frac{p+5}{p+1} \\
1000 &= \frac{p+5}{p+1} \\
1000(p+1) &= p+5 \\
1000p + 1000 &= p+5 \\
999p &= -995 \\
p &= -\frac{995}{999} \approx -1
\end{aligned}$$

Example 3: Equating the arguments. Solve for x

$$\begin{aligned}
\log{(2x^2 - 7x - 4)} &= \log{(2x+16)} \\
2x^2 - 7x - 4 &= 2x + 16 \\
2x^2 - 9x - 20 &= 0 \\
(2x+5)(x-4) &= 0 \\
x &= -\frac{5}{2} \text{ or } x = 4
\end{aligned}$$

6.7 More Equations

Example 1: An exponential with a product in it

$$7(1.06^x) = 5.2 \quad \leftrightarrow \quad 1.06^x = \frac{5.2}{7} \quad \leftrightarrow \quad x = \frac{\log\left(\frac{5.2}{7}\right)}{\log 1.06} \approx -5.68$$

Example 2: Logarithms where an answer doesn't work out

$$\log_6 x + \log_6 (x+1) = 1 \implies \log_6 [x(x+1)] = 1x(x+1) = 6^1 = 6$$
$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x = -3 \text{ or } x = 2$$

Note: $x = -3$ is not a valid solution as logarithms of negative numbers are undefined.

6.8 The Logarithmic Scale in Science

6.8.1 A. Earthquakes

The Logarithm Scale in the Physical Sciences

The Richter Scale defines the magnitude of an earthquake as:

$$M = \log\left(\frac{I}{I_0}\right)$$

Where I is the earthquake intensity measured and I_0 is the intensity of a reference quake.

Example 1: Comparing Earthquake Magnitudes

The California earthquake that interrupted the World Series in 1989 measured 6.9 on the Richter scale. The quake that caused the 2004 tsunami in Indonesia measured 9.2. How much more powerful was the Indonesian quake?
Intuitively: The difference in magnitude is $9.2 - 6.9 = 2.3$.
Using the Definition:

$$10^{9.2}/10^{6.9} = 10^{2.3} \approx 200$$

The Indonesian earthquake was about 200 times more powerful.

6.8.2 B. Sound

The decibel scale compares sound intensities:

$$L = 10\log\left(\frac{I}{I_0}\right)$$

Where I is the intensity of the sound being measured and I_0 is the threshold of human hearing (the quietest sound we can hear).

Example 2: Decibel Calculation

A sound is 5000 times more intense than one that is just audible. How many decibels is the sound?

$$L = 10 \log(5000) \approx 10 \times 3.699 \approx 37$$

Example 3: Comparing Loudness

A jet engine emits a 160 dB sound, while Niagara Falls is 90 dB. How many times louder than Niagara Falls is a jet engine?

$$10^{160/10}/10^{90/10} = 10^{16-9} = 10^7$$

A jet engine is 10^7 times louder than Niagara Falls.

6.8.3 C. The pH Scale

The acidity or alkalinity of a solution is given by the pH scale:

$$\text{pH} = -\log[H^+]$$

Where $[H^+]$ is the concentration of hydronium ions in the solution in mol/L.

Example 4: Calculating pH

The hydronium ions in blood measure at a concentration of 4×10^{-7} mol/L. What is the pH of blood?

$\text{pH} = -\log(4 \times 10^{-7}) = -(\log 4 + \log 10^{-7}) = -(\log 4 - 7) \approx -0.602 - 7 = 6.4$

Example 5: Finding Hydronium Ion Concentration

What is the concentration of hydronium ions in a pool if the pH is 8.2?

$$8.2 = -\log[H^+] \quad \leftrightarrow \quad [H^+] = 10^{-8.2} \approx 6.3 \times 10^{-9} \text{ mol/L}$$