Grade 11 Functions Notes

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Contents

1 Unit 1: Intro To Functions

1.1 Lesson 1 - Domain and Range

The domain and range of a function describe the possible input and output values, respectively.

Example 1: For $f(x) = \sqrt{x}$, the domain is $x \geq 0$ since you can't take the square root of a negative number without delving into complex numbers.

Example 2: For $f(x) = \frac{1}{x}$, the domain is $x \neq 0$ since division by zero is undefined.

Example 3: For a parabola $f(x) = x^2$, the range is $f(x) \geq 0$.

Lesson 2 - Function Notation

Function notation introduces a more concise way to represent equations.

Example 1: Given $f(x) = 2x^2+3$, find $f(2)$. Solution: $f(2) = 2(2^2)+3 = 11$.

Example 2: If $f(x) = x + 5$, what is $f(3)$? Solution: $f(3) = 3 + 5 = 8$.

Function Table for $f(x) = x^2$:

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

1.2 Lesson 3 - Max/Min of Quadratics

The vertex of a quadratic function indicates its maximum or minimum value.

Properties of Quadratic Expressions

Quadratic expressions, depending on their leading coefficients, have certain inherent properties:

1. Any square of a number, x^2, is always non-negative. Thus, $x^2 \geq 0$. This is because squaring any real number, whether positive or negative, results in a positive value (or zero if $x = 0$).

2. The negative of a square, $-x^2$, is always non-positive. Thus, $-x^2 \leq 0$. This is the opposite behavior of x^2, as negating it ensures the parabola opens downward.

3. For a quadratic in the form of $-(x-h)^2$, where h is a constant, the expression represents a downward-opening parabola shifted h units to the right on the x-axis. As an example, for $-(x-4)^2$, the parabola is shifted 4 units to the right, and $-(x-4)^2 \leq 0$.

Note: The sign and nature of the leading coefficient in a quadratic expression can give insights into the orientation of the parabola and its range.

Example 1: Find the vertex of $f(x) = 2x^2 + 4x + 3$. By completing the square, the vertex is $(-1, 1)$.

Example 2: For $f(x) = -x^2 + 4x - 3$, the vertex is $(2, 5)$ and represents a maximum due to the negative leading coefficient.

Lesson 4 - Radicals

Radicals involve taking roots of numbers.

Example 1: Simplify $\sqrt[3]{27}$. Solution: 3.

Example 2: Simplify $\sqrt{81}$. Solution: 9.

Example 3: Determine the domain of $f(x) = \sqrt{x-5}$. Solution: $x \geq 5$.

Lesson 5 - Solve Quadratics by Factoring

Factoring is a method to solve quadratic equations.

Example 1: Solve $x^2 - 5x + 6 = 0$. Solution: $(x-2)(x-3) = 0$, so $x = 2$ or $x = 3$.

Example 2: Solve $x^2 - x - 6 = 0$. Solution: $(x-3)(x+2) = 0$, so $x = 3$ or $x = -2$.

1.3 Lesson 6 - Quadratic Formula

When factoring is not possible, the quadratic formula offers a solution.

Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Solve $x^2+x-1=0$. Plugging the coefficients into the formula will give two solutions for x.

1.4 Lesson 7 - Linear Quadratic Systems

Linear Quadratic Systems involve a combination of linear and quadratic equations. Solving these systems can reveal points of intersection between the two functions, if they exist.

Methods of Solution:

1. **Substitution:** Use the linear equation to solve for y (or x), and then substitute this expression into the quadratic equation.
2. **Graphical:** Graph both the linear and quadratic functions on the same set of axes and identify the point(s) of intersection.

Example 1: Solve the system:

$$y = x^2 + 2$$
$$y = 2x + 3$$

By substitution, set $x^2 + 2$ equal to $2x + 3$. Solving this equation will give the x-coordinates of the intersection points. To find the corresponding y-coordinates, substitute these x-values into either the linear or quadratic equation.

Example 2: Solve the system:

$$y = x^2 - 4$$
$$y = -x + 2$$

Again, using substitution, equate $x^2 - 4$ to $-x + 2$. This will yield the x-coordinates of where the line intersects the parabola. To find the corresponding y-values, plug these x-values into one of the original equations.

Note: Sometimes, a linear function might not intersect a quadratic function, or it might intersect at one or two points. The nature of intersection can also be discerned graphically or by assessing the discriminant when setting the two equations equal to each other.

1.5 Translations and Function Notation

Given a function $y = f(x)$, the following transformations can be applied:

- **Vertical Translation:** $y = f(x) + c$ shifts the graph c units upward (if $c > 0$) or downward (if $c < 0$).
- **Horizontal Translation:** $y = f(x-h)$ shifts the graph h units to the right (if $h > 0$) or to the left (if $h < 0$).
- **Vertical Stretch/Compression:** $y = af(x)$ stretches the graph by a factor of a if $a > 1$, or compresses it if $0 < a < 1$. If $a < 0$, the graph is also reflected about the x-axis.
- **Horizontal Stretch/Compression:** $y = f(bx)$ compresses the graph horizontally by a factor of b if $b > 1$, or stretches it if $0 < b < 1$.

A translation is a type of transformation that changes the location of a function in the coordinate plane, while preserving its shape and size.

1.5.1 Representation Using Function Notation

Translations can be represented using function notation:

$$y = f(x - h) + k$$

This represents the function $y = f(x)$ translated horizontally by h units and vertically by k units.

- If $h > 0$, the function is translated to the right.
- If $h < 0$, the function is translated to the left.
- If $k > 0$, the function is translated up.
- If $k < 0$, the function is translated down.

Sketching Translated Graphs

To sketch the graph of $y = f(x - h) + k$, start with the graph of $f(x)$ and translate points on that function based on the values of h and k. Asymptotes, if any, must also be translated.

Translations of Common Base Functions

Base Function	Translated Function
$f(x) = x^2$	$y = (x-h)^2 + k$
$f(x) = \sqrt{x}$	$y = \sqrt{x-h} + k$
$f(x) = \frac{1}{x}$	$y = \frac{1}{x-h} + k$

★ Domain and Range

When a function is translated, the domain and range of the function are translated as well.

Examples

Example 1: Consider the function $y = \sqrt{x}$.

Transformation: The graph of $y = 2\sqrt{x-3} + 1$:

- Starts with the basic square root graph.
- Stretches vertically by a factor of 2.
- Translates 3 units to the right.
- Translates 1 unit upwards.

Description: This graph will resemble the basic upward curving square root graph, but will be steeper (due to the vertical stretch), and shifted to the point (3,1) as its starting point.

Example 2: Consider the function $y = x^2$.

Transformation: The graph of $y = -0.5(x+2)^2 - 4$:

- Starts with the basic parabolic graph.
- Reflects about the x-axis (due to the negative sign).
- Compresses vertically by a factor of 0.5.
- Translates 2 units to the left.
- Translates 4 units downward.

Description: This graph will resemble an upside-down parabola, opening downward, being wider than the standard $y = x^2$ graph (due to the vertical compression), and having its vertex at the point (-2,-4).

Shortcut Words For Transformation

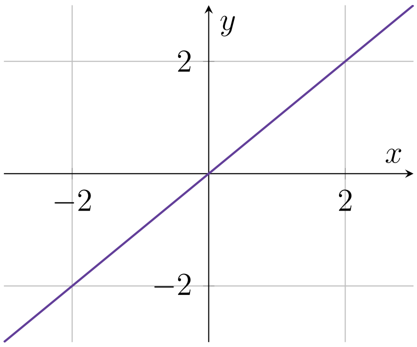
VT $\rightarrow$ Vertical Translation
HT $\rightarrow$ Horizontal Translation
VS $\rightarrow$ Vertical Stretch
HS $\rightarrow$ Horizontal Stretch
RXA $\rightarrow$ Reflection in x-axis
RYA $\rightarrow$ Reflection in y-axis

Summary Lesson

Notation	Transformation Type	Coordinate Change
$f(x)+d$	Vertical translation up d units	$(x,y)\mapsto(x,y+d)$
$f(x)-d$	Vertical translation down d units	$(x,y)\mapsto(x,y-d)$
$f(x+c)$	Horizontal translation left c units	$(x,y)\mapsto(x-c,y)$
$f(x-c)$	Horizontal translation right c units	$(x,y)\mapsto(x+c,y)$
$-f(x)$	Reflection over x-axis	$(x,y)\mapsto(x,-y)$
$f(-x)$	Reflection over y-axis	$(x,y)\mapsto(-x,y)$
$af(x)$	Vertical stretch for $\lvert a\rvert>1$	$(x,y)\mapsto(x,ay)$
$af(x)$	Vertical compression for $\lvert a\rvert<1$	$(x,y)\mapsto(x,ay)$
$f(bx)$	Horizontal compression for $\lvert b\rvert>1$	$(x,y)\mapsto\left(\frac{x}{b},y\right)$
$f(bx)$	Horizontal stretch for $\lvert b\rvert<1$	$(x,y)\mapsto\left(\frac{x}{b},y\right)$

Parent Functions

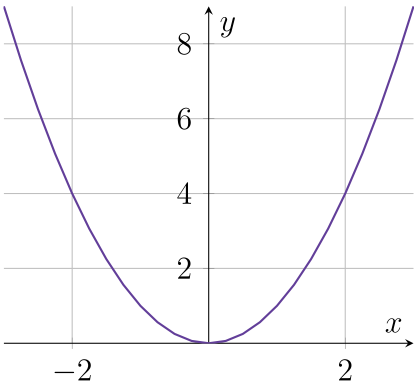
1. Linear Function: $f(x) = x$



x	$f(x)$
-2	-2
-1	-1
0	0
1	1
2	2

The linear function $f(x) = x$ represents a straight line that passes through the origin (0,0) and has a slope of 1. As x increases or decreases, $f(x)$ increases or decreases respectively.

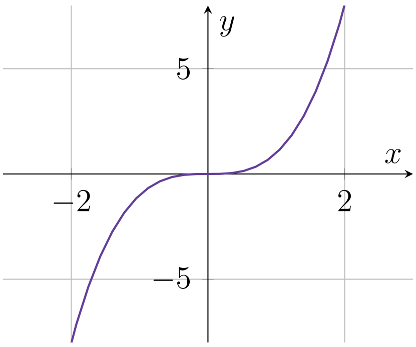
2. Quadratic Function: $f(x) = x^2$



x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

The quadratic function $f(x) = x^2$ represents a parabola that opens upwards and has its vertex at the origin (0,0). The function values are always non-negative and increase quadratically as x moves away from 0.

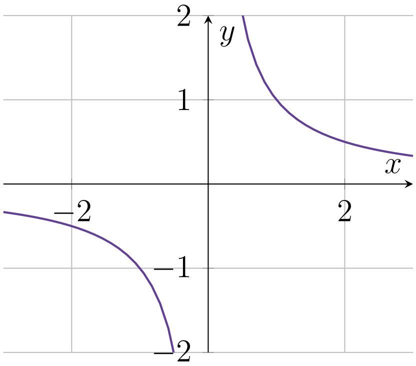
3. Cubic Function: $f(x) = x^3$



x	$f(x)$
-2	-8
-1	-1
0	0
1	1
2	8

The cubic function $f(x) = x^3$ has a characteristic S-shape and crosses the origin (0,0). The function values increase cubically as x moves away from 0, with $f(x)$ being negative when x is negative and positive when x is positive.

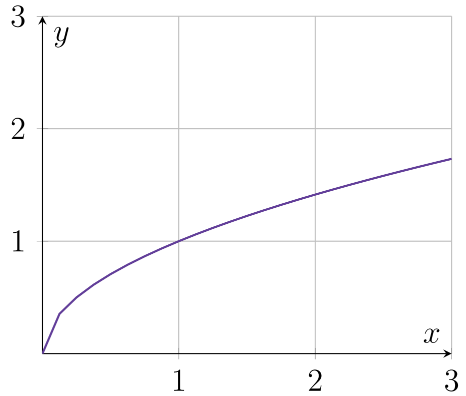
4. Reciprocal Function: $f(x) = \frac{1}{x}$



x	$f(x)$
-3	-0.33
-2	-0.5
-1	-1
1	1
2	0.5
3	0.33

The reciprocal function $f(x) = \frac{1}{x}$ has two hyperbolas in the 1st and 3rd quadrants. As x approaches 0 from either side, $f(x)$ approaches $\pm\infty$.

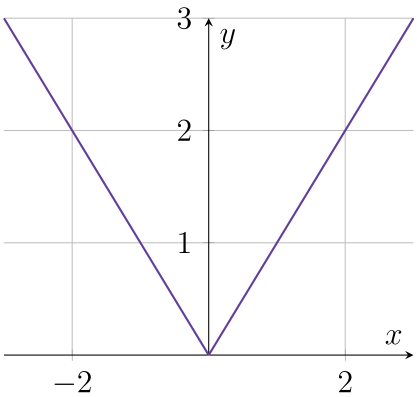
5. Square Root Function: $f(x) = \sqrt{x}$



x	$f(x)$
0	0
1	1
2	1.41
3	1.73

The square root function $f(x) = \sqrt{x}$ is defined for $x \geq 0$ and represents half of a parabola that opens upwards. As x increases, $f(x)$ increases more slowly.

6. Absolute Value Function: $f(x) = |x|$



x	$f(x)$
-2	2
-1	1
0	0
1	1
2	2

The absolute value function $f(x) = |x|$ represents a V-shaped graph that has its vertex at the origin (0,0). The function values are always non-negative, regardless of the sign of x.

2 Unit 2: Rational Expressions

Lesson 1 - Review of Exponent Rules

The exponent rules are foundational principles that dictate how terms with the same base can be combined.

1. $a^m \times a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{m \times n}$

Example:

1. Using Rule 1: $2^3 \times 2^4 = 2^{3+4} = 2^7$
2. Using Rule 2: $\frac{5^7}{5^4} = 5^{7-4} = 5^3$
3. Using Rule 3: $(3^2)^3 = 3^{2 \times 3} = 3^6$

Lesson 2 - Rational Exponents

Rational exponents refer to exponents that are fractions. They can often be represented as roots.

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Example:

1. Using the formula: $9^{\frac{1}{2}} = \sqrt[2]{9} = 3$
2. $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$
3. $8^{\frac{2}{3}} = \sqrt[3]{8^2} = 4$

Lesson 3 - Simplifying, Multiplying and Dividing Rational Expressions

Rational expressions are fractions wherein either the numerator, the denominator, or both are polynomials.

1. To multiply: Multiply the numerators with each other and the denominators with each other.
2. To divide: Multiply the first fraction by the reciprocal of the second.

Example:

1. Multiplication: $\frac{x}{y} \times \frac{z}{w} = \frac{x \times z}{y \times w}$
2. Division: $\frac{x}{y} \boldsymbol{\nabla} \cdot \frac{z}{w} = \frac{x}{y} \times \frac{w}{z}$
3. Simplifying: $\frac{3x}{6y} = \frac{x}{2y}$ Divided by 2

Lesson 4 - Adding and Subtracting Rational Expressions

To add or subtract rational expressions:

1. Find a common denominator.
2. Rewrite each fraction with that denominator.
3. Add or subtract the numerators.

Example:

1. $\frac{a}{c} + \frac{b}{d} = \frac{ad+bc}{cd}$ given that cd is the common denominator.
2. $\frac{3x}{x^2-1} + \frac{2x}{x^2+2x} = \frac{3x(x+2)+2x(x-1)}{x^2-1}$
3. $\frac{5}{x+3} - \frac{2}{x-2} = \frac{5(x-2)-2(x+3)}{(x+3)(x-2)}$

Factoring Review

Factoring is the process of expressing a polynomial as a product of simpler polynomials. This document will demonstrate how to factor polynomials with examples and step-by-step explanations.

Factoring Basics

To factor a polynomial, we look for common factors and apply various factoring techniques. Here are some common factoring methods:

1. Factoring out the greatest common factor (GCF).
2. Factoring by grouping.
3. Factoring the difference of squares.
4. Factoring trinomials of the form $ax^2 + bx + c$.
5. Factoring special forms like the sum or difference of cubes.

Examples

Example:

Factoring the GCF.

Factor the polynomial $6x^2 + 12x$.

$$6x^2 + 12x = 6x(x + 2) \quad \text{(Factor out the GCF, 6x)}$$

Example:

Factoring by Grouping.

Factor the polynomial $x^3 - x^2 + 4x - 4$.

$$\begin{aligned} x^3 - x^2 + 4x - 4 &= (x^3 - x^2) + (4x - 4) \quad \text{(Group the terms)} \\ &= x^2(x - 1) + 4(x - 1) \quad \text{(Factor out common factors)} \\ &= (x^2 + 4)(x - 1) \quad \text{(Factor further if possible)} \end{aligned}$$

Example:

Factoring the Difference of Squares.
Factor the polynomial $9y^2 - 16z^2$.

$$\begin{aligned} 9y^2 - 16z^2 &= (3y)^2 - (4z)^2 \quad \text{(Recognize it as a difference of squares)} \\ &= (3y + 4z)(3y - 4z) \quad \text{(Apply the difference of squares formula)} \end{aligned}$$

Example:

Factoring a Trinomial.
Factor the trinomial $x^2 + 5x + 6$.

$$x^2 + 5x + 6 = (x + 2)(x + 3) \quad \text{(Find two numbers that multiply to 6 and add up to 5)}$$

Example:

Factoring the Sum of Cubes.
Factor the polynomial $x^3 + 8$.

$$\begin{aligned} x^3 + 8 &= (x + 2)(x^2 - 2x + 4) \quad \text{(Recognize it as a sum of cubes)} \\ &= (x + 2)(x - 1 + 2i)(x - 1 - 2i) \quad \text{(Factor the quadratic using the quadratic formula)} \end{aligned}$$

3 Unit 3: Quadratic Functions

Quadratic functions are a class of polynomial functions of the form $f(x) = ax^2 + bx + c$, where a, b, and c are constants, and a is not equal to zero. They play a crucial role in algebra, calculus, physics, engineering, and various other fields.

3.1 The Standard Form of Quadratic Functions

A quadratic function is typically expressed in standard form as:

$$f(x) = ax^2 + bx + c$$

Here is a brief explanation of the parameters:

- a: The coefficient of the quadratic term. It determines the direction in which the parabola opens (upwards if $a > 0$, and downwards if $a < 0$).
- b: The coefficient of the linear term. It shifts the vertex of the parabola horizontally.
- c: The constant term. It shifts the vertex of the parabola vertically.

3.2 Vertex Form of a Quadratic Function

The vertex form of a quadratic function is particularly useful for identifying the vertex and other properties. It is expressed as:

$$f(x) = a(x - h)^2 + k \tag{1}$$

In this form, the vertex of the parabola is represented by the point (h, k).

3.3 Vertex and Axis of Symmetry

The vertex of a quadratic function in standard form ($f(x) = ax^2 + bx + c$) can be determined using the following formulas:

$$x_{\text{vertex}} = \frac{-b}{2a} \tag{2}$$

$$y_{\text{vertex}} = f(x_{\text{vertex}}) \tag{3}$$

In vertex form ($f(x) = a(x-h)^2+k$), the vertex is already given as (h, k).

The axis of symmetry is a vertical line that passes through the vertex. It is given by the equation:

$$x = \frac{-b}{2a} \tag{4}$$

3.4 Discriminant and Solutions

Quadratic functions may have real or complex solutions. The discriminant (Δ) can be used to determine the nature of the solutions:

$$\Delta = b^2 - 4ac \tag{5}$$

The solutions are classified as follows:

- If $\Delta > 0$, the function has two distinct real solutions.
- If $\Delta = 0$, the function has one real solution (a repeated root).
- If $\Delta < 0$, the function has two complex solutions.

3.5 Graph of a Quadratic Function

A graphical representation of a quadratic function helps us visualize its behavior. Let's consider an example:

$$f(x) = 2x^2 - 3x + 1 \tag{6}$$

3.6 Vertex Calculation

Using the formulas, we can find the vertex:

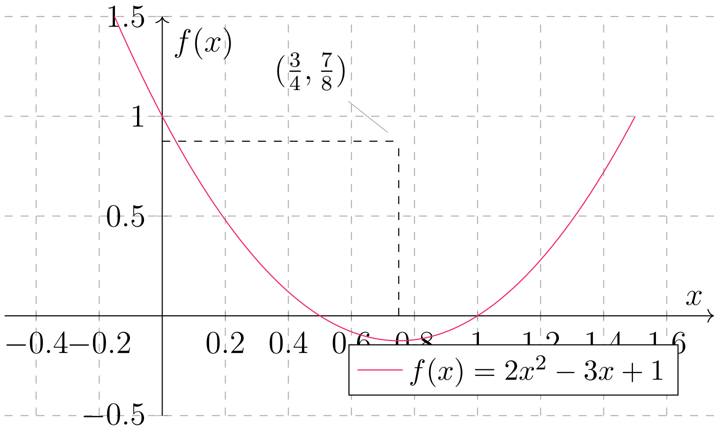
$$x_{\text{vertex}} = \frac{-(-3)}{2(2)} = \frac{3}{4}$$

$$y_{\text{vertex}} = f\left(\frac{3}{4}\right) = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 1 = \frac{7}{8}$$

So, the vertex of the quadratic function is $\left(\frac{3}{4}, \frac{7}{8}\right)$.

3.7 Graph

You can visualize the graph of the quadratic function:



Remember!

Parabolas (Standard Form)

Equation	$y = ax^2 + bx + c$
Vertex	$(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})$
Opening Direction	Down if $a > 0$
Y-intercept	$(0, c)$

Parabolas (Vertex Form)

Equation	$y = a(x-h)^2 + k$
Vertex	(h, k)
Opening Direction	Up if $a > 0$, Down if $a < 0$

3.8 Applications of Quadratic Functions

Quadratic functions are not merely theoretical; they have numerous practical applications in various fields. Some common applications include:

1. **Physics**: Quadratic functions describe the motion of objects under the influence of gravity. The equation $h(t) = -16t^2 + v_0 t + h_0$ models the height (h) of an object at time (t) when it is thrown vertically with an initial velocity (v_0) from an initial height (h_0).

2. **Engineering**: In structural engineering, quadratic equations model the deformation of materials under load, helping engineers design stable structures.

3. **Economics**: Quadratic functions are used to model cost, revenue, and profit functions in business and economics. These functions assist in optimizing production and pricing strategies.

4. **Computer Graphics**: In computer graphics, quadratic functions are used to create smooth curves and surfaces. For instance, Bézier curves are defined using quadratic equations.

5. **Biology**: Quadratic functions can model population growth or decline of species. The logistic growth model is an example of such an application.

6. **Statistics**: In regression analysis, quadratic functions are used to model complex relationships between variables.

7. **Astronomy**: Quadratic equations can describe the orbits of celestial bodies and the motion of planets.

Note:

Quadratic functions are versatile and play a fundamental role in mathematics and various scientific disciplines. Understanding their properties, equations, and applications is crucial to problem solving and modeling real-world phenomena. Whether in physics, engineering, economics, or any other field, the knowledge of quadratic functions is a valuable asset in tackling complex problems.

4 Unit 4: Exponential Functions

4.1 Radicals

Parts of radicals

$$\sqrt[n]{a}$$

- n = index or root
- a = Radicand

PROPERTIES OF RADICALS

a. $a^{\frac{1}{n}} = \sqrt[n]{a}$

b. $a^{\frac{m}{n}} = \sqrt[n]{a} = (\sqrt[n]{a})^m$

c. $\sqrt[n]{a^n} = a^{\frac{n}{n}}$

d. $\sqrt[n]{ab} \cdot \sqrt[n]{b}$

Example:

1. $x^{\frac{1}{3}} = \sqrt[3]{x}$
2. $x^{\frac{2}{3}} = \sqrt[3]{x^2}$ or $(\sqrt[3]{x})^2$
3. $\sqrt{x^2} = x \quad \sqrt[5]{x^5} = x$

Example:

1. $\sqrt{36y^4} = \sqrt{36} \cdot \sqrt{y^4} = 6y^2$
2. 2. $\sqrt{72y^5} = \sqrt{36y^4} \cdot \sqrt{2y} = 6y^2\sqrt{2y}$
3. $\sqrt[3]{48y^7} = \sqrt[3]{8y^6} \cdot \sqrt[3]{6y} = 2y^2\sqrt[3]{6y}$
4. $\sqrt[4]{64x^5y^8} = \sqrt[4]{16x^4y^8}\sqrt[4]{4x} = 2xy^2\sqrt[4]{4x}$
5. $\sqrt[5]{64x^5y^8} = \sqrt[5]{32x^5y^5} \cdot \sqrt[5]{2y^5} = 2xy^5\sqrt{2y^5}$
6. $\sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}$
7. 7. $\sqrt[3]{\frac{8y^4}{27x^3}} = \frac{\sqrt[3]{8y^4}}{\sqrt[3]{27x^3}} = \frac{\sqrt[3]{8y^3} \cdot \sqrt[3]{y}}{\sqrt[3]{27x^3}} = \frac{2y^3\sqrt{y}}{3x}$
8. $\sqrt{\frac{x^2}{4y^2}} = \frac{\sqrt{x^2}}{\sqrt{4y^2}} = \frac{x}{2y}$

Rationalizing the Denominator

When simplifying fractions with radicals, you need to rationalize the denominator by multiplying the numerator and the denominator by the **smallest value that will allow you to eliminate the the radical in the denominator**, as shown below.

Example:

1. $\sqrt{\frac{1}{5}} = \frac{\sqrt{1}}{\sqrt{5}} = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{25}} = \frac{\sqrt{5}}{5}$
2. $\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$
3. $\sqrt[3]{\frac{1}{x}} = \frac{\sqrt[3]{1}}{\sqrt[3]{x}} = \frac{1}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{\sqrt[3]{x^2}}{x}$
4. $\sqrt[4]{\frac{4p^8}{8p^8}} = \sqrt[4]{\frac{p^2}{2}} = \frac{\sqrt[4]{p^2}}{\sqrt[4]{2}} \cdot \frac{\sqrt[4]{2^3}}{\sqrt[4]{2^3}} = \frac{\sqrt[4]{8p^2}}{2}$

Note:

Rules for Simplifying Radicals:

1. There should be no factor in the radicand that has a power greater than or equal to the index.
2. There should be no fractions under the radical sign.
3. There should be no radicals in the denominator (i.e. the denominator should be rationalized).

ADDITION AND SUBTRACTION

Radicals may be added or subtracted when they have the same index and the same radicand(just like combining like terms).

Note:
When adding or subtracting radicals, the index and radicand do not change.

Example:

1. $5\sqrt{2} - 8\sqrt{2} = -3\sqrt{2}$
2. $6x\sqrt[3]{3} + 2x\sqrt[3]{3} = 8x\sqrt[3]{3}$
3. $5\sqrt[5]{xy} + 6\sqrt[5]{xy} = 11\sqrt[5]{xy}$
4. $7\sqrt{x} - 9\sqrt[3]{x} + 4\sqrt[3]{x} = 7\sqrt{x} - 5\sqrt[3]{x}$
5. $\sqrt{75} + 2\sqrt{12} - 5\sqrt{3} = \sqrt{25}\sqrt{3} + 2\sqrt{4}\sqrt{3} - 5\sqrt{3} + 4\sqrt{3} - 5\sqrt{5} = 4\sqrt{3}$

MULTIPLICATION OF RADICALS

To multiple radicals, just multiply using the same rules as multiplying polynomials (Distributive Property, FOIL, and Exponent Rules) except **NEVER** multiply values outside of the radicals times values inside the radical.

Example:

1. $\sqrt{20x^3} \cdot \sqrt{4xy^6} = \sqrt{80x^4y^7} = \sqrt{16x^4y^6} \cdot \sqrt{5y} = 4x^2y^3\sqrt{5y}$
2. $2x\sqrt{3xy} \cdot 4\sqrt{2x^5y}$
3. $2\sqrt{5}(3\sqrt{2} - \sqrt{5}) = 6\sqrt{10} - 2\sqrt{25} = 6\sqrt{10} - 10$
4. $(2\sqrt{x} + 2)(\sqrt{x} + 3) = 2\sqrt{x^2} + 6\sqrt{x} + 2\sqrt{x} + 6$

Note:

When multiplying radicals with different indexes, change to rational exponents first, find a common denominator in order to add the exponents, then rewrite in radical notation as shown below:
Example: $\sqrt[3]{x^2} \cdot \sqrt[6]{x^5} = x^{\frac{2}{3}} \cdot x^{\frac{5}{6}} = x^{\frac{4}{6}} \cdot x^{\frac{5}{6}} = x^{\frac{3}{2}} = \sqrt{x^3} = \sqrt{x^2}\sqrt{x} = x\sqrt{x}$

MORE RATIONALIZING THE DENOMINATOR: (DIVISION)

If the denominator contain two terms such that at least one term has a radical, multiply the numerator and the denominator by the **conjugate** of the denominator: **Conjugate** - the conjugate of a binomial of the form (a+b) is (a-b). Example: The conjugate of $(\sqrt{x}-3) is (\sqrt{x}+3)$.

Note:

Since $(a+b)(a-b) = a^2 - b^2$, eliminating the middle term, multiplying by the conjugate eliminates the middle term that would still have a radical in it, thus removing the radical from the denominator.

Example:

$$a. \frac{1}{\sqrt{x}+1} \cdot \frac{\sqrt{x}-1}{\sqrt{x}-1} = \frac{\sqrt{x}-1}{\sqrt{x^2}-1} = \frac{\sqrt{x}-1}{x-1}$$

$$b. \frac{6}{\sqrt{5}-\sqrt{2}} \cdot \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}} = \frac{6(\sqrt{5}+\sqrt{2})}{5-2} = \frac{6(\sqrt{5}+\sqrt{2})}{3} = 2(\sqrt{5}+\sqrt{2})$$

4.2 Exponent Laws

Exponent Laws

Product Law

When multiplying two terms with the same base, add the exponents.

$$a^m \cdot a^n = a^{m+n}$$

Quotient Law

When dividing two terms with the same base, subtract the exponents.

$$\frac{a^m}{a^n} = a^{m-n}$$

Power Law

When raising a power to another power, multiply the exponents.

$$(a^m)^n = a^{mn}$$

Zero Exponent Law

Any nonzero number raised to the power of zero is equal to 1.

$$a^0 = 1$$

Negative Exponent Law

$$a^{-n} = \frac{1}{a^n}$$

Exponent Laws

1. $x^3 \cdot x^4 = x^{3+4} = x^7$
2. $\frac{y^6}{y^3} = y^{6-3} = y^3$
3. $a^5 \cdot a^{-2} = a^{5-2} = a^3$
4. $\frac{b^8}{b^4} = b^{8-4} = b^4$

4.3 Logarithms

Logarithms

Introduction to Logarithms

Logarithms are the inverse operations of exponentiation.

Properties of Logarithms

a. $\log_b(a \cdot c) = \log_b(a) + \log_b(c)$

b. $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$

c. $\log_b(a^n) = n \cdot \log_b(a)$

Common Logarithm and Natural Logarithm

a. Common logarithm: $\log_{10}(x) = \log(x)$

b. Natural logarithm: $\ln(x)$

Examples

Logarithms

1. $10^x = 100$ implies $x = 2$
2. $\log_2(8) = 3$ because $2^3 = 8$
3. $e^y = 20$ implies $y = \ln(20)$
4. $\ln(1) = 0$ because $e^0 = 1$

Solving equations with logarithms

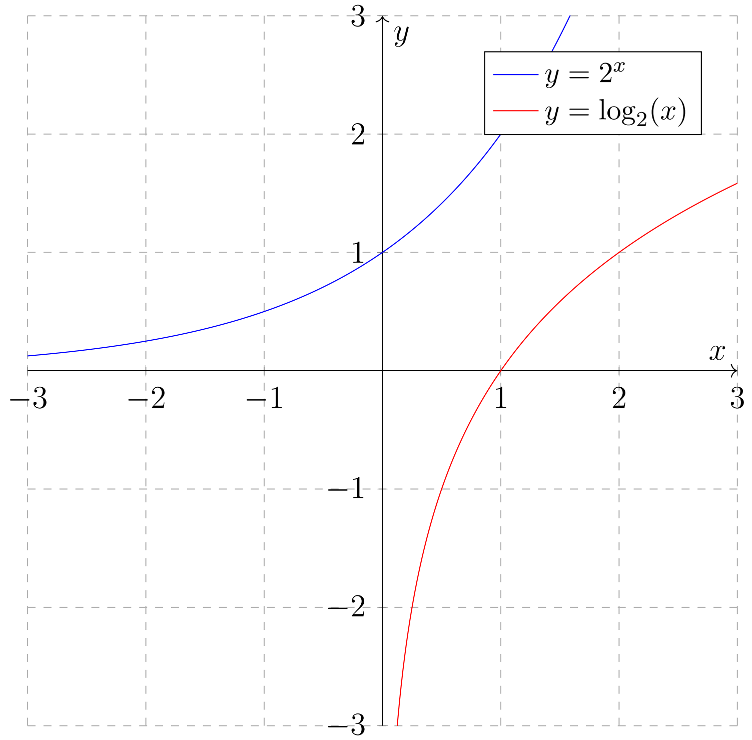
⇒ The logarithms of a # to given base is the exponent that must be used with that base to obtain the given #.

Example:
The logarithms of 64 to base 2 is 6 since $2^6 = 64$

We would write that as: $\log_2 64 = 6$.

- $\log_2$ - base.
- 64 - argument.
- 6 - logarithms.

⇒ The logarithm function is the $\underbrace{\textbf{inverse}}_{\text{(Switch } x \text{ and } y)}$ function of an exponent.



$\Rightarrow$ In general we write $y = \log_a x$ where a is the base and x is the argument

Example:

Determine the logs:

a. $\log_{10} 1000 = 3$ since $10^3 = 1000$

b. $\log_5 625 = 5^x = 625 = x = 4$

c. $\log_2 1024 = 10$

d. $\log_b b = 1$

e. $\log_2 1 = 0$

Example:

Express in exponential form:

1. $m = \log_3 81 \Rightarrow 3^m = 81$
2. $y = \log_7(\frac{1}{7}) \Rightarrow 7^y = \frac{1}{7}$
3. $n = \log_a n \Rightarrow a^m = n$

Example:

Express as logarithm

1. $2^5 = 32 \Rightarrow 5 = \log_2 32$
2. $3^m = 343 \Rightarrow m = \log_3 343$
3. $\frac{1}{25} = 5^x \Rightarrow n = \log_5(\frac{1}{25})$

Example:

Use your calculator to Evaluate

1. $\log_6 216 \Rightarrow 3$
2. $\log_7 117649 \Rightarrow \frac{\log 117649}{\log 7} = 6$
3. $\log 1000000 \Rightarrow \text{base } 10(\text{common log}) = 6$

Log law

1. $\log a^x \Rightarrow x \log a$

2. $a^x = (10^{\log a})^x \Rightarrow a^x = 10^{x \log a}$

Example:

1. $12^x = 400 \Rightarrow x = \log_1 2400 = x = \frac{\log 400}{log12} = x \approx 2.41$
2. $1.08^x = 4.39 \Rightarrow \log 10.8^x = \log 4.39 = x = \frac{log4.39}{1.08} = x \approx 19.22$
3. $10^{x-3} = 500$

$$
\begin{aligned}
10^{x-3} &= 500 \\
\Rightarrow \log 10^{x-3} &= \log 500 \\
(x-3)\log 10 &= \log 500 \\
x - 3 &= \frac{\log 500}{\log 10} \\
x - 3 &= \frac{\log 500}{\log 10} \\
x &= 3 + \frac{\log 500}{\log 10} \\
x &\approx 5.70
\end{aligned}
$$

4.4 Transformation of Exponential Function

$$\underbrace{f(x) = a \cdot b^{k(k-d)} + C}_{\text{base of exponential fn}}$$

Exponential functions can be transformed in the same way as $f(x) = a \cdot b^{k(k-d)} + C$ other function. The graph of can be found by performing transformations on the graph of y = bx

Example:
List transformation applied to $y = 2^x$

1. $f(x) = 3 \cdot 2^x - 5$
   - VS by 3
   - Parent function $y = 2^x$
   - VT 5 ↓
2. $f(x) = -2^{x-1} + 6$
   - RXA
   - HT 1 →
   - VT 6 ↑
3. $f(x) = \frac{1}{2}(2)^{-x+4} - 8$
   - VS by 2
   - RXA
   - HT 4 →
   - VT 6 ↑
4. $f(x) = -2^{3x-9} + 62$
   - RXA
   - HS by $\frac{1}{3}$
   - VT 62 ↑

Example:

List the the transformation applied to $y = \left(\frac{1}{4}\right)^x$

1. $f(x) = 3\left(\frac{1}{4}\right)^{x-10} - 8$
   - VS
   - HT 10 $\rightarrow$
   - VT 8 $\downarrow$
2. $g(x) = -\left(\frac{1}{4}\right)^{\frac{1}{2}x} + 2$
   - RXA
   - HS by 2
   - VT 2 $\downarrow$
3. $h(x) = \frac{1}{3}\left(\frac{1}{4}\right)^{-x-6}$
   - VS by $\frac{1}{3}$
   - RYA
   - HT 6 $\leftarrow$
4. $p(x) = -3\left(\frac{1}{4}\right)^{2x+1}$
   - RXA
   - VS by 3
   - HT by $\frac{1}{2}$
   - HS by $\frac{1}{2}$

For an exponential function, the horizontal asymptote is only affected by vertical translation. So the equation of the H.A will be $y = c$

When the function is:

$$f(x) = a \cdot b^{k(x-d)} + C$$

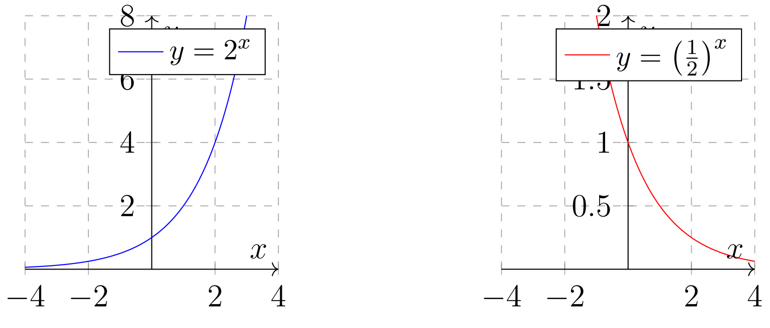
Example:

Find the equation of the horizontal asymptote and determine the y-intercept (set $x = 0$).

1. $f(x) = 2 \cdot 3^x - 4$ H.A $= -4$ y-int is -2
2. $g(x) = \frac{1}{2}\left(\frac{1}{8}\right)^{-x} + 1$ H.A $= 1$ y-int is $\frac{3}{2}$

Exponential Functions

⇒ The domain of exponential function is always: $\{x \in \mathbb{R}\}$.



(a) Growth (b) Decay

Figure 1: Growth and Decay

⇒ The range will depend on the location of the horizontal asymptote and if there was a reflection in the x-axis (RXA).

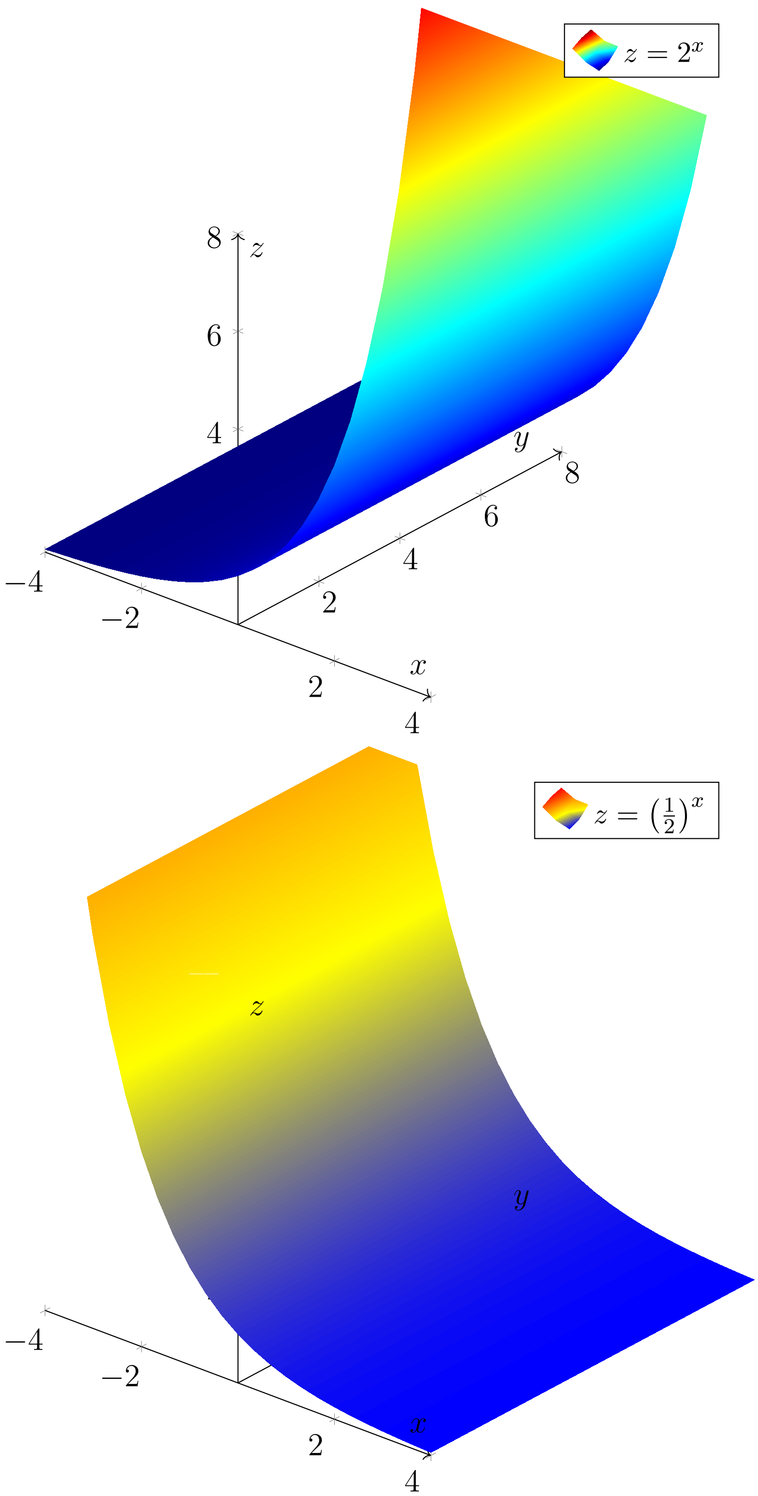
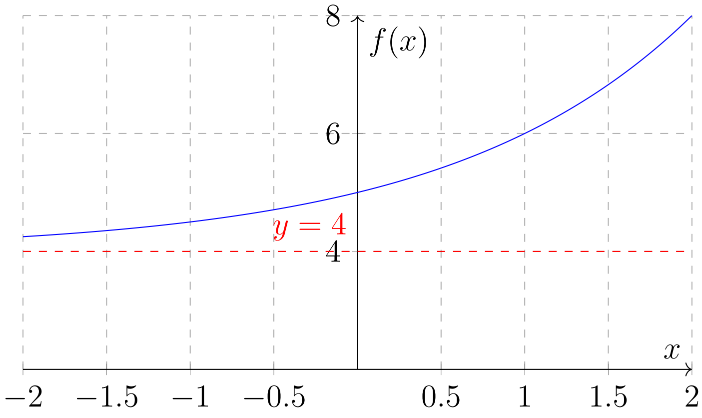


Figure 2: Growth and Decay in 3D Plot

Examples

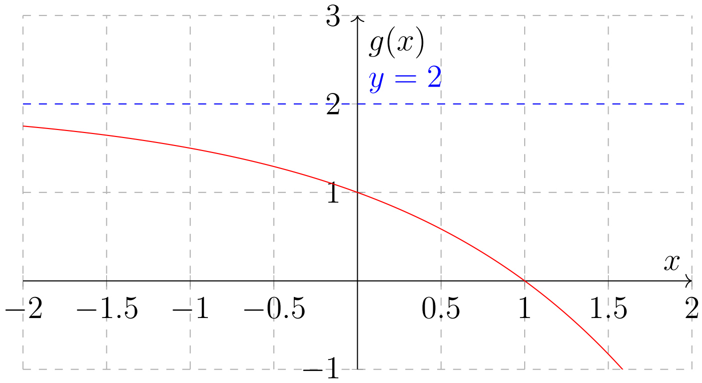
Example:

1. $f(x) = 2^x + 4$



$\{y \in \mathbb{R} \mid y > 4\}$

2. $g(x) = -2^x + 2$



$\{y \in \mathbb{R} \mid y < 2\}$

Example:

Ex.1: For the function, find:

1. Parent function
2. Horiz asymptote
3. y-int
4. Transformations
5. Domain & Range

a. $f(x) = 4^{2(x+5)} - 8$
   Parent function: 4^x
   Transformation:

   (a) HS by $\frac{1}{2}$
   (b) HT 5 $\leftarrow$
   (c) VT 8 $\downarrow$

   y-int:

$$
\begin{aligned}
f(0) &= 4^{2(0+5)} - 8 \\
&= 4^{10} - 8 \\
&= 1048568
\end{aligned}
$$

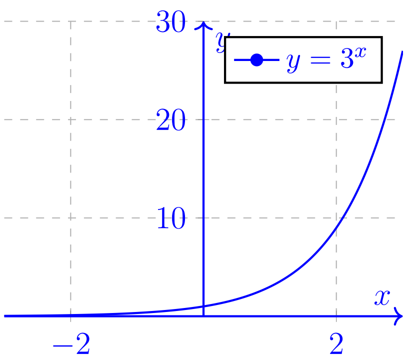
   H.A: $y = -8$
   Domain: $\{x \in \mathbb{R}\}$ Range: $\{y \in \mathbb{R} \mid y > -8\}$

Graphing Exponential Functions

Ex.1: The function $y = 3^x$ is an exponential function because the exponent is a variable.

Now, let's look at how to graph the exponential function $y = 3^x$.

x	$y = 3^x$	y	(x, y)
-3	$3^{-3} = \frac{1}{3^3}$	$\frac{1}{27}$	$(-3, \frac{1}{27})$
-2	$3^{-2} = \frac{1}{3^2}$	$\frac{1}{9}$	$(-2, \frac{1}{9})$
-1	$3^{-1} = \frac{1}{3^1}$	$\frac{1}{3}$	$(-1, \frac{1}{3})$
0	$3^0 = 1$	1	$(0, 1)$
1	$3^1 = 3$	3	$(1, 3)$
2	$3^2 = 9$	9	$(2, 9)$
3	$3^3 = 27$	27	$(3, 27)$



Definition 1: Since the y values increase as the x values increase in the example above, this is what we call exponential **Growth**. (The graph goes up the hill from left to right)

QUESTION: In the exponential function $y = 3^x$, the y=values can never equal or be less than **zero**.

Definition 2: Since the y-values can NEVER equal to zero in this function, there is a horizontal asymptote at $y = 0$.

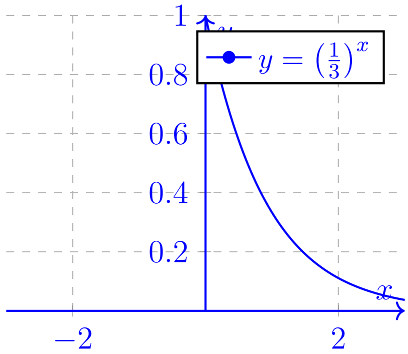
Ex.2: By looking at the graph above, list the domain and range of the function $y = 3^x$.

Domain:$\{x \in \mathbb{R}\}$. This is because the function is defined for every real value of x.

Range: $\{y \in \mathbb{R} \mid y > 0\}$. This is evident from the graph, where the values of y are positive for all corresponding values of x.

Ex.3: Consider the function $y = \left(\frac{1}{3}\right)^x$. Analyze the graph and determine its domain and range.

x	$y = \left(\frac{1}{3}\right)^x$	y	(x, y)
-3	27	27	$(-3, 27)$
-2	9	9	$(-2, 9)$
-1	3	3	$(-1, 3)$
0	1	1	$(0, 1)$
1	$\frac{1}{3}$	$\frac{1}{3}$	$(1, \frac{1}{3})$
2	$\frac{1}{9}$	$\frac{1}{9}$	$(2, \frac{1}{9})$
3	$\frac{1}{27}$	$\frac{1}{27}$	$(3, \frac{1}{27})$



Definition 2: Since the y-values decrease as the x-values increase in the example above, this is what we call exponential decay. (The graph goes down the hill from left to right).

QUESTION: Is there an asymptote? If so, where it is?

Yes , it is on "y=0".

Ex.4: By looking at the graph above, list the domain and range of the function $y = (\frac{1}{3})^x$.

Domain:$\{x \in \mathbb{R}\}$. This is because the function is defined for every real value of x.

Range: $\{y \in \mathbb{R} \mid y > 0\}$. This is evident from the graph, where the values of y are positive for all corresponding values of x.

Ex.3: Consider the function $y = \left(\frac{1}{3}\right)^x$. Analyze the graph and determine its domain and range.

1 - Exponential Growth

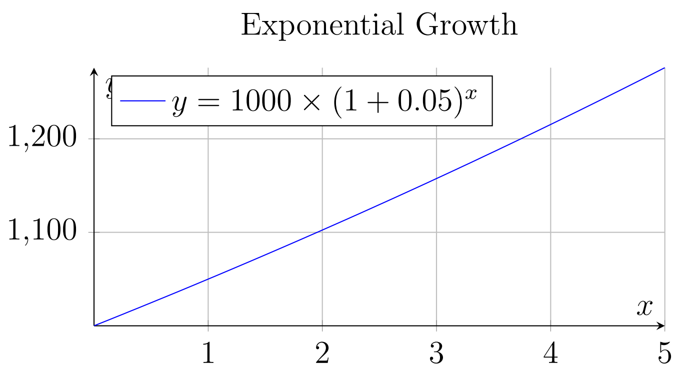
Exponential growth is a captivating concept where a quantity increases at a fixed percentage rate over time. This growth is modeled by the formula $y = ab^x$, where a is the initial amount, b is the growth factor, and x is the time variable.

Example:

Suppose you invest 1000 at an annual interest rate of 5%, compounded annually. The growth formula is $A = 1000 \times (1 + 0.05)^x$. After 3 years, the amount would be approximately $A = 1000 \times (1 + 0.05)^3 \approx 1157.63$.

Note:

The graph of an exponential growth function is characterized by a distinct upward curve that becomes steeper as b increases.

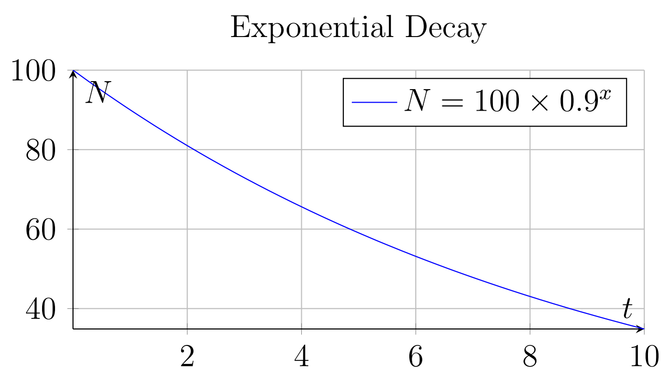


2 - Exponential Decay

Exponential decay is the counterpart to exponential growth. It occurs when a quantity decreases at a fixed percentage rate over time. The decay is modeled by the formula $y = ab^x$, where b is between 0 and 1.

Example:
Consider a radioactive substance that decays at a rate of 10% per year. Its decay formula is $N = N_0 \times 0.9^t$. After 5 years, the remaining quantity is $N = N_0 \times 0.9^5$.

Note:
The graph of an exponential decay function exhibits a decreasing curve that approaches but never reaches zero.



3 - Compound Interest

Compound interest is a powerful concept where interest is added to the initial principal, which then earns interest over time. The compound interest formula is given by $A = P(1 + r/n)^{nt}$, where A is the final amount, P is the principal, r is the annual interest rate, n is the number of times interest is compounded per year, and t is the time in years.

Example:
Imagine investing 5000 at an annual interest rate of 6%, compounded quarterly. The formula is $A = 5000 \times (1 + 0.06/4)^{4t}$. After 2 years, the amount is $A = 5000 \times (1 + 0.06/4)^{4\times 2}$.

Note:
Compound interest enables your investment to grow faster compared to simple interest, especially with more frequent compounding.

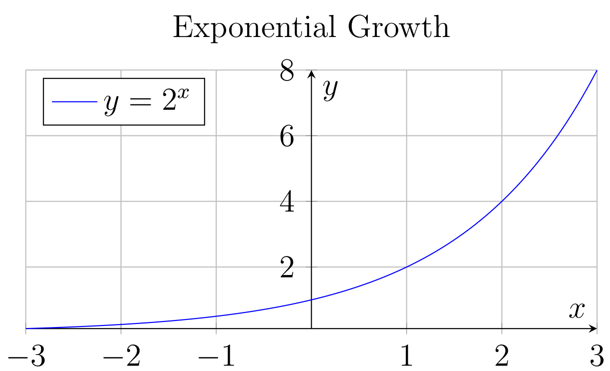
4 - Properties of Exponential Functions

Exponential functions possess several key properties:

- They have a constant base.
- They can model growth or decay.
- They have an asymptote, which they approach but never reach.
- They are always positive if the base is greater than 1.
- They are always decreasing if the base is between 0 and 1.

Example:
Consider the function $f(x) = 2^x$. It has a constant base of 2 and models exponential growth.

Note:
The graph of an exponential function approaches but never crosses the horizontal axis (asymptote).

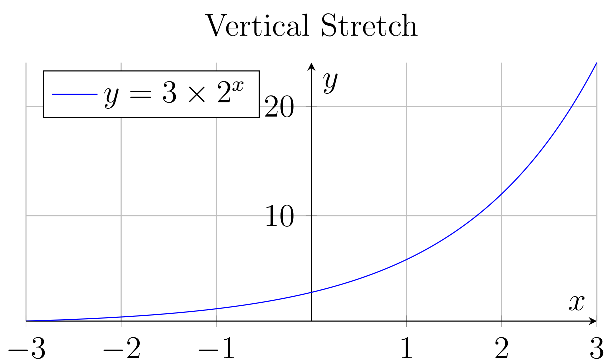


5 - Transformations

Transformations offer a way to modify the graph of an exponential function. Common transformations include vertical shifts, horizontal shifts, reflections, and stretches or compressions. These transformations are applied to the base function $y = b^x$.

Example:
If $g(x) = 3 \times 2^x$, the function g is a vertical stretch of $f(x) = 2^x$ by a factor of 3.

Note:
Transformations alter the appearance and behavior of the exponential function graph.



Transformations can also involve horizontal shifts, reflections, and other modifications to customize the behavior of the exponential function graph.

6 - Applications of Exponential Functions

$$A = P(1+i)^n$$

Where:

- A is the final amount
- P is the initial amount
- i is the growth/decay rate
- n is the total number of growth/decay periods

Doubling Times: An increase of 100% (or 1) which makes the base equal to 2.

$$A = P(1+i)^n$$
$$\therefore A = P(2)^n$$

Half-Lives: A decrease of 50%(or 0.5) which makes the base equal to $\frac{1}{2}$.

$$A = P(1-0.5)^n$$

$$\therefore A = P(0.5)$$

or

$$A = P(\frac{1}{2})^n$$

Example:
1.The element Duzzanium has a half-life of 4 months. If there are 5000 g of Duzzanium today, how much will there be in 2 years?

Example:
2. A bacterial culture began with 7500 bacteria. It's growth can be modeled using the formula $N = 7500(2)^{\frac{t}{36}}$, where N is the number of bacteria after t hours.

a. What is the doubling time of the bacteria?

b. How many bacteria are present after 36 hours?

c. How many bacteria are present after 3 days?

d. Approximately how many hours will pass for the culture to reach 2 million bacteria?

Summative Assessment

1. **Evaluate the following:**

   a) 2^3:

$$\begin{aligned} 2^3 &= 2 \times 2 \times 2 \\ &= 8 \end{aligned}$$

   b) 10^{-2}:

$$\begin{aligned} 10^{-2} &= \frac{1}{10^2} \\ &= \frac{1}{100} \\ &= 0.01 \end{aligned}$$

   c) e^0:

$$e^0 = 1$$

2. **Solve for x:**

   a) $5^x = 125$:

$$\begin{aligned} 5^x &= 125 \\ x &= 3 \end{aligned}$$

   b) $2e^{2x} = 16$:

$$\begin{aligned} e^{2x} &= 8 \\ 2x &= \ln(8) \\ x &= \frac{\ln(8)}{2} \end{aligned}$$

3. **Consider the function $f(x) = 3 \times 2^x$. Perform the following transformations and sketch the resulting graph:**

   a) **Vertical stretch by a factor of 2:** The function becomes $g(x) = 6 \times 2^x$.

   b) **Horizontal shift right by 1 unit:** The function becomes $h(x) = 3 \times 2^{(x-1)}$.

   c) **Reflection across the x-axis:** The function becomes $k(x) = -3 \times 2^x$.

   Graph: (Note: This is a conceptual sketch; precise plotting requires numerical values.)

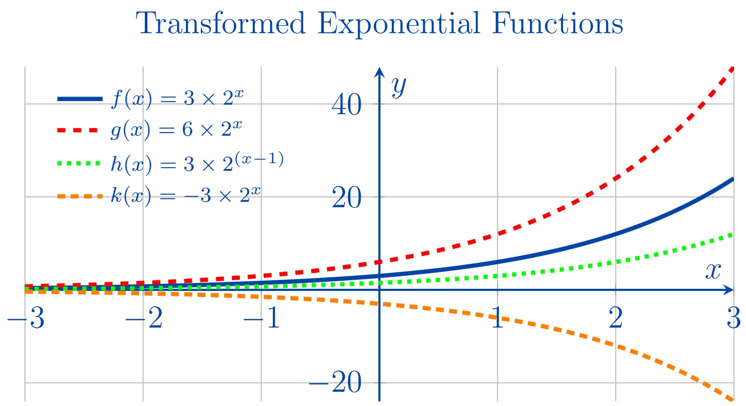


Figure 3: Transformed Exponential Functions

Note:

When considering the function $y = a^x$:

Exponential Growth: If $a > 1$, the function exhibits exponential growth. In this case, as x increases, the corresponding values of y grow rapidly.

Exponential Decay: If $0 < a < 1$, the function demonstrates exponential decay. In such instances, as x increases, the values of y diminish rapidly, showcasing a decay behavior.

5 Unit 5: Trignometric Ratios

5.1 Basic Geometry

In this section, we will explore fundamental concepts in geometry.

Euclidean Geometry Euclidean geometry is the study of flat space.

Theorem 5.1. *The sum of angles in a triangle is always* $180°$*.*

Proof. This follows from the parallel postulate. □

5.2 Trigonometry

Now, let's delve into trigonometry.

5.3 Trigonometric Functions

Trigonometric functions relate angles to the sides of a right triangle.

Definition 5.1. *The sine function, denoted* sin*, is defined as the ratio of the opposite side to the hypotenuse.*

Example:
For a right triangle with an angle of $30°$, if the opposite side is 3 and the hypotenuse is 6, then $\sin(30°) = \frac{3}{6} = \frac{1}{2}$.

5.4 Primary Trigonometric Ratios

The primary trigonometric ratios in a right triangle are defined as follows:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

Example: Consider a right triangle with an angle θ such that $\sin(\theta) = \frac{3}{5}$. Find $\cos(\theta)$ and $\tan(\theta)$.

Solution: Using the fact that $\sin^2(\theta) + \cos^2(\theta) = 1$, we can find $\cos(\theta)$:

$$\cos^2(\theta) = 1 - \sin^2(\theta)$$
$$\cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$$

Since θ is in the first quadrant, $\cos(\theta) = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$. Now, use the definition of $\tan(\theta)$ to find $\tan(\theta)$:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{3/5}{4/5} = \frac{3}{4}$$

5.5 Reciprocal Trigonometric Ratios

The reciprocal trigonometric ratios are defined as the reciprocals of the primary trigonometric ratios:

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

Example: If $\sec(\theta) = \frac{5}{3}$, find $\sin(\theta)$.

Solution: Since $\sec(\theta) = \frac{1}{\cos(\theta)}$, we can find $\cos(\theta)$ first:

$$\cos(\theta) = \frac{1}{\sec(\theta)} = \frac{3}{5}$$

Now, use the definition of $\sin(\theta)$:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{5^2 - 3^2}}{5} = \frac{4}{5}$$

5.6 Solving Right Triangles

To solve a right triangle, you need to find the lengths of all sides and the measures of all angles. Use the primary and reciprocal trigonometric ratios to relate the angles and side lengths.

Example: In a right triangle, if $\sin(\alpha) = \frac{4}{5}$, find $\cos(\alpha)$ and $\tan(\alpha)$.

Solution: Using the fact that $\cos(\alpha) = \sqrt{1 - \sin^2(\alpha)}$ and $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$, we can calculate:

$$\cos(\alpha) = \frac{3}{5}, \quad \tan(\alpha) = \frac{4}{3}$$

5.7 Solving Oblique Triangles

For oblique triangles (non-right triangles), the Law of Sines and Law of Cosines are used:

5.8 Sine Law

The Law of Sines states that for any triangle:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

Example: In triangle ABC, $a = 8$, $b = 11$, and $\angle C = 35°$. Find the length of side c.

Solution: Using the Law of Sines, we have:

$$\frac{\sin(C)}{c} = \frac{\sin(A)}{a} \implies c = \frac{\sin(C) \cdot a}{\sin(A)}$$

Substitute the given values to find c.

5.9 Cosine Law

The Law of Cosines relates the lengths of the sides of a triangle to the cosine of one of its angles:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Example: In triangle XYZ, $x = 7$, $y = 9$, and $\angle Z = 120°$. Find the length of side z.

Solution: Using the Law of Cosines, we have:

$$z^2 = x^2 + y^2 - 2xy\cos(Z)$$

Substitute the given values to find z.

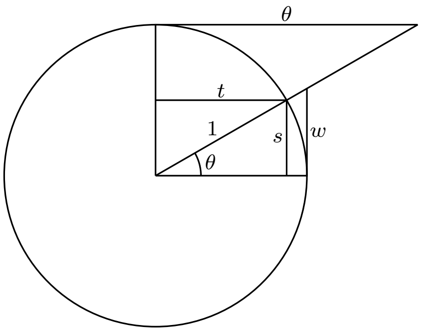
5.10 Challenging Problems

Problem 1: In triangle PQR, $p = 10$, $q = 15$, and $\angle R = 45°$. Find the lengths of sides r and s.

Problem 2: In triangle LMN, $l = 12$, $\angle M = 30°$, and $\angle N = 105°$. Find the lengths of sides m and n.

Problem 3: In triangle ABC, $a = 6$, $b = 8$, and $\angle C = 90°$. Find the lengths of sides c and d, where d is the altitude from $\angle C$ to side AB.

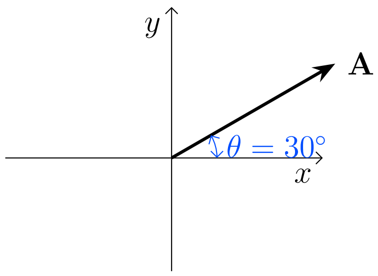
5.11 Trigonometric equation



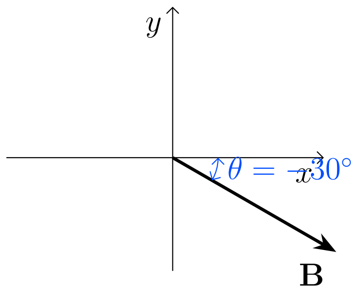
Pythagorean identity
$\sin^2\theta + \cos^2\theta = 1$
$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\tan\theta =$ Slope.

Angles in standard position means 0° is the position x-axis and positive angles move counter-clockwise, negative angles move clockwise.

Ex.1: Show the following angles:



(a)

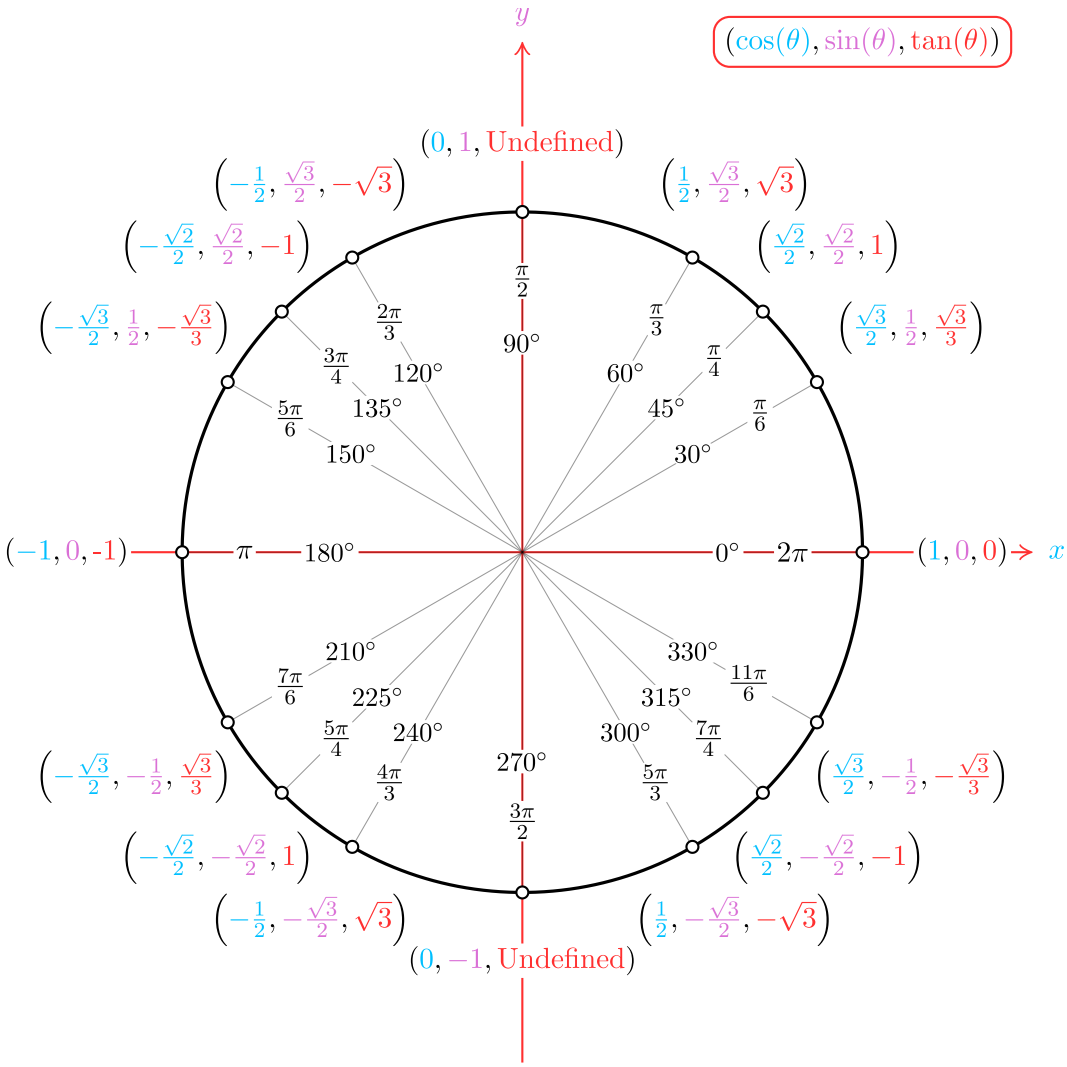


(h)

Coterminal angles: Angles in standard position that have the same terminal arms.

For example: $30°, 390°$ and $-330°$ are all coterminaz angles.

5.12 Unit Circle

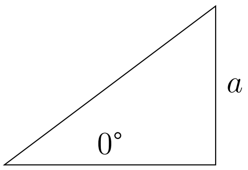


5.13 Trigonometric Ratios for Special Angles

5.14 Special Angles

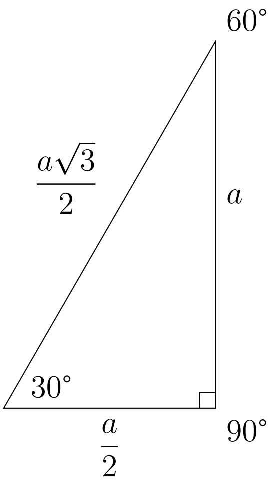
In trigonometry, certain angles have special significance due to their simplicity and exact values. The primary special angles are 0°, 30°, 45°, 60°, and 90°.

0° (Zero Degrees)



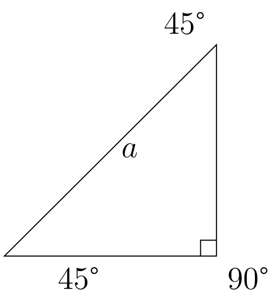
$$\sin(0°) = 0,$$
$$\cos(0°) = 1,$$
$$\tan(0°) = 0.$$

30°



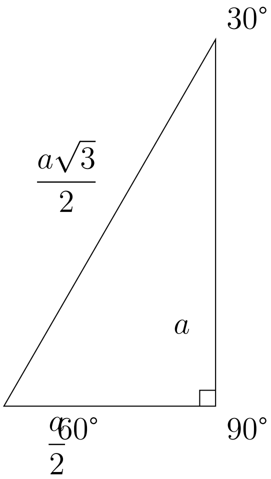
$$\sin(30°) = \frac{1}{2},$$
$$\cos(30°) = \frac{\sqrt{3}}{2},$$
$$\tan(30°) = \frac{1}{\sqrt{3}}.$$

45°



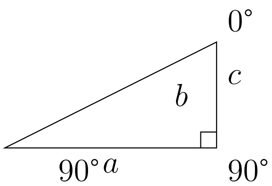
$$\sin(45°) = \frac{\sqrt{2}}{2},$$
$$\cos(45°) = \frac{\sqrt{2}}{2},$$
$$\tan(45°) = 1.$$

60°



$$\sin(60^\circ) = \frac{\sqrt{3}}{2},$$
$$\cos(60^\circ) = \frac{1}{2},$$
$$\tan(60^\circ) = \sqrt{3}.$$

90°



$$\sin(90^\circ) = 1,$$
$$\cos(90^\circ) = 0\,(\text{undefined}),$$
$$\tan(90^\circ) = \infty\,(\text{undefined}).$$

5.15 Coterminal Angles:

Coterminal angles are angles that share the same initial and terminal sides but can differ by integer multiples of a full revolution (360° or 2π radians). Two angles θ and $\theta + 360n$ (where n is an integer) are coterminal.

5.16 Principal Angles:

The principal angle is the smallest positive angle between the terminal side of an angle and the x-axis. For any angle θ, the principal angle θ_p is given by:

$$\theta_p = \theta - 360n$$

5.17 Trig identities

An identity is an equation which is true for all values of the variable.

5.18 Reciprocal identity

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

5.19 Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

Rearranging Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$
$$\cos^2\theta = 1 - \sin^2\theta$$
$$\sin^2\theta = 1 - \cos^2\theta$$

5.20 Quotient identity:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

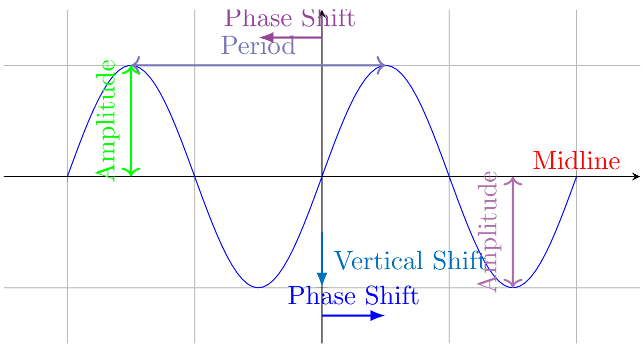
Steps to prove identities

1. Simplify one side at a time (Both sides is not allowed.)
2. Start with more complicated side first.
3. Simplify one side as much as you can, if you get stuck than switch to take other side.
4. Converting everything into sine and cos is sometimes helpful.
5. Use your intuition!

6 Unit 6: Sinusoidal Functions

Sinusoidal function are periodic function where graph looks like smooth symmetical waves, where any potion can be horizontally translated onto another potion of curve.
Graph of sinusdidal function can be created by transforming $y = \sin\theta$ and $y = \cos\theta$.

A periodic function is a function that repeats its values in regular intervals. In this document, we will explore the properties and examples of periodic functions.
Note: sinusoidal function are all period function but period function are not sinusoidal.



Midline, amplitude, and period are three features of sinusoidal graphs.

6.1 Characteristics of Sinusoidal Graphs

6.2 Midline/Axis of the Curve:

The midline is a horizontal line right in the middle between the highest and lowest points of the graph. It's calculated using the formula:

$$y = \frac{\text{max value} + \text{min value}}{2}$$

6.3 Amplitude:

The amplitude is how high or low the graph reaches from the middle line. You find it using:

$$a = \frac{\text{max value - min value}}{2}$$

6.4 Period:

The period is how wide one complete cycle of the graph is. It's found by looking at the distance between two consecutive high or low points. The formula is:

$$P = \frac{2\pi}{k} \quad \text{or} \quad P = \frac{360}{k}$$

6.5 Phase Shift:

The phase shift is like a sideways movement of the graph, showing if it's shifted left or right. For $y = A\sin(k\theta + d) + C$, you calculate it using:

$$\text{Horizontal shift} = \frac{d}{k} \text{ or } -\frac{d}{k}$$

6.6 Vertical Shift:

The vertical shift is like lifting or lowering the whole graph. For $y = A\sin(k\theta + d) + C$, it shifts the entire graph up or down by C units.

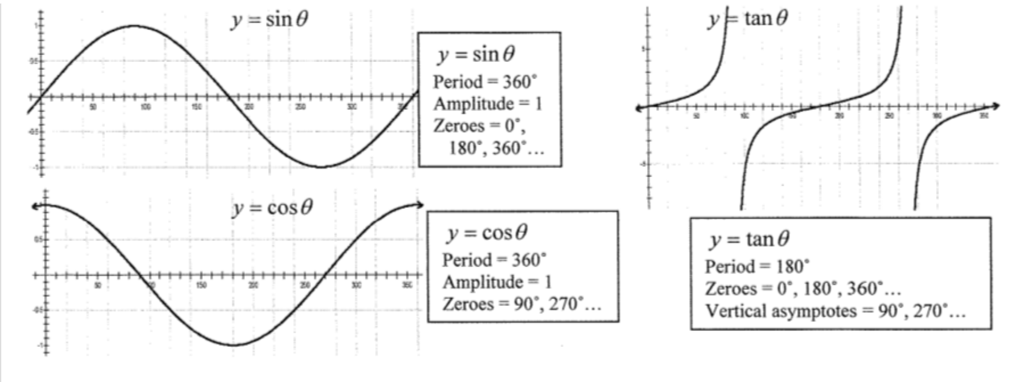
6.7 Key Intervals:

Key intervals are specific ranges of values where the graph undergoes significant changes. These intervals are essential for identifying crucial points, such as the highest and lowest values, contributing to a better understanding of the function's behavior.

One important key interval is $\frac{\text{Period}}{4}$, representing a quarter of the period. It marks a position in the graph where distinctive shifts and changes occur, aiding in the analysis of the function's characteristics.

6.8 Trigonometric Functions

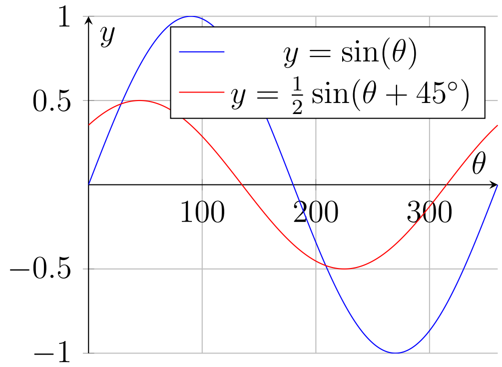
The graphs of $y = \sin\theta, y = \cos\theta, y = \tan\theta$. are shown below.



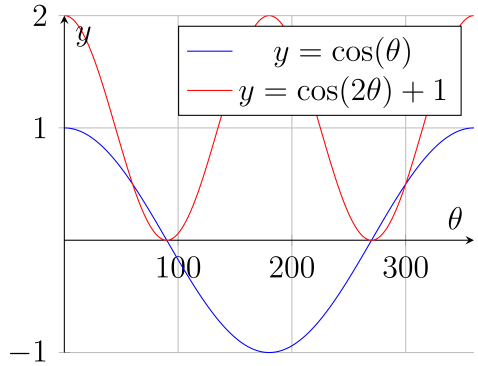
6.9 Transformations of Trigonometric Functions

- Transformations apply to trig functions as they do to any other function.
- The graphs of $y = a\sin k(\theta + d) + c$ and $y = a\cos k(\theta + b) + d$ are transformations of the graphs $y = \sin\theta$ and $y = \cos\theta$ respectively.
- The value of a determines the vertical stretch, called the amplitude. It also tells whether the curve is reflected in the θ-axis.
- The value of k determines the horizontal stretch. The graph is stretched by a factor of $\frac{1}{k}$. We can use this value to determine the period of the transformation of $y = \sin\theta$ or $y = \cos\theta$.
- The period of $y = \sin k\theta$ or $y = \cos k\theta$ is $\frac{360°}{k}, k > 0$. The period of $y = \tan k\theta$ is $\frac{180°}{k}, k > 0$.
- The value of d determines the horizontal translation, known as the phase shift.
- The value of c determines the vertical translation. $y = d$ is the equation of the axis of the curve.

Examples

- 

1
0.5
−0.5
−1
y
θ
100
200
300
$y = \sin(\theta)$
$y = \frac{1}{2}\sin(\theta + 45°)$

- 

2
1
−1
y
θ
100
200
300
$y = \cos(\theta)$
$y = \cos(2\theta) + 1$

6.10 Property of Sine function

Property	$y = \sin(x)$
Amplitude	1
Period	360°
Equation of Axis	$y = 0$
Domain	$\{x \mid x \in \mathbb{R}\}$
Range	$\{y \in \mathbb{R} \mid -1 \leq y \leq 1\}$
x-intercepts	$x \in \{180n, n \in \mathbb{Z}\}$
y-intercept	0
Maximum	$1,$ when $x \in \{90 + 360n, n \in \mathbb{Z}\}$
Minimum	$-1,$ when $x \in \{270 + 360n, n \in \mathbb{Z}\}$
Intervals of increase	$90 < x < 270,$ and all intervals obtained by adding $360n, n \in \mathbb{Z}$
Intervals of decrease	$270 < x < 90,$ and all intervals obtained by adding $360n, n \in \mathbb{Z}$

6.11 Property of Cosine function

Property	$y = \cos(x)$
Amplitude	1
Period	360°
Equation of Axis	$y = 0$
Domain	$\{x \| x \in \mathbb{R}\}$
Range	$\{y \in \mathbb{R} \| -1 \leq y \leq 1\}$
x-intercepts	$x \in \{90 + 180n, n \in \mathbb{Z}\}$
y-intercept	1
Maximum	$1,$ when $x \in \{360n, n \in \mathbb{Z}\}$
Minimum	$-1,$ when $x \in \{180 + 360n, n \in \mathbb{Z}\}$
Intervals of increase	$0 < x < 180,$ and all intervals obtained by adding $360n, n \in \mathbb{Z}$
Intervals of decrease	$180 < x < 360,$ and all intervals obtained by adding $360n, n \in \mathbb{Z}$

Extra

Definition

A function $f(x)$ is periodic with period T if, for all x in the domain of f, the following holds:

$$f(x+T) = f(x)$$

This means that the function values repeat every T units along the x-axis.

Examples

Sine Function

The sine function, denoted by $\sin(x)$, is a classic example of a periodic function. Its period is 2π, and the function is defined as:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

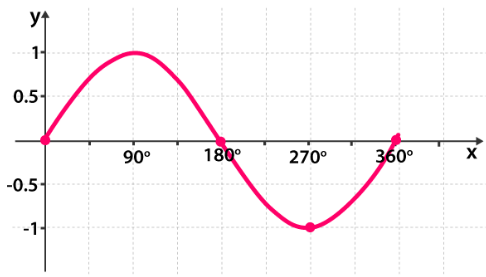


Figure 4: Graph of the sine function.

Square Wave

The square wave is another example of a periodic function. It has a period T and is defined as:

$$\text{square wave}(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{T}{2} \\ -1 & \text{if } \frac{T}{2} \leq x < T \end{cases}$$

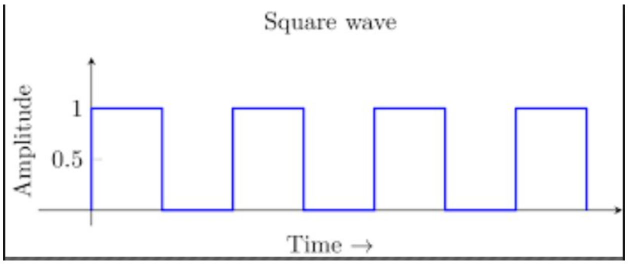
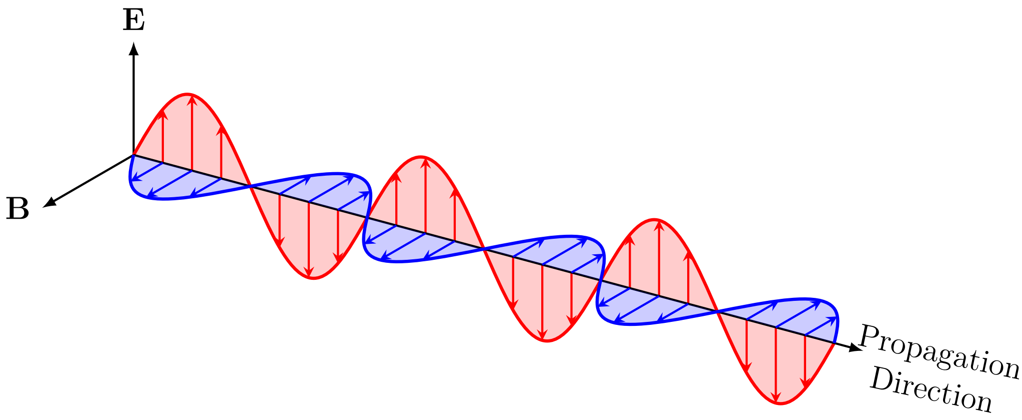


Figure 5: Graph of a square wave.

Conclusion

Periodic functions are essential in various branches of mathematics and physics. Understanding their properties and behavior is crucial for analyzing and modeling periodic phenomena.



7 Unit 7: Discrete Functions (Series and Sequences)

7.1 Arithmetic sequence

Discrete functions - Sequences and series

⇒ Sequence - An ordered list of numbers (e.g., $2, 5, 8, 11, \ldots$)

⇒ Term - A number in a sequence (e.g., t_1 = first term, t_{10} = tenth term)

⇒ Arithmetic sequence ⇒ A sequence that has the same difference (d) between any pair of consecutive terms.

Example:

$$\underbrace{2, 5, 8, 11, \ldots}_{\text{Common difference of 3}}$$

⇒ General term - A formula, labeled t_n, that expresses each term of a sequence as a function of its position.

Example:
$2, 4, 6, 8, 10, \ldots$ has a general term of $t_n = 2n$.

$$t_1 = 2(1) = 2$$
$$t_4 = 2(4) = 8$$
$$t_{24} = 2(24) = 48$$

⇒ The general term for an arithmetic sequence is:

$$t_n = a + (n-1)d$$

Where:

- a is the first term.
- d is the common difference
- n is the term #

Example:

Ex.1:

a. Find the general term for the arithmetic sequence:

b. Find t_9 (the 9^{th} term)

i) $10, 14, 18, 22, \ldots$

a.

$$\text{Given: } \quad a = 10 \text{ (first term)}$$
$$t_n = 10 + (n-1)(4)$$

b.

$$\begin{aligned} t_9 &= a + (9-1)d \\ &= 10 + 8(4) \\ &= 42 \end{aligned}$$
$$\boxed{\therefore\ t_9 = 42}$$

ii) $-33, -23, -13, -3, 7, \ldots$

a.

$$\text{Given: } \quad a = -33 \text{ (first term)}$$
$$\begin{aligned} d &= t_2 - t_1 \quad \text{(common difference)} \\ &= -23 - (-33) \\ &= 10 \end{aligned}$$
$$\begin{aligned} t_n &= -33 + (n-1)(10) \\ &= 10n - 43 \end{aligned}$$

b.

$$\begin{aligned} t_9 &= -33 + (9-1)(10) \\ &= -33 + 8(10) \\ &= -33 + 80 \\ &= 47 \\ &\boxed{\therefore\ t_9 = 47} \end{aligned}$$

Example:

Ex.2: Find the 33^{3rd} term in the sequence 18, 11, 4, -3, ...
Solution:

$$\begin{aligned} a &= 10 \\ d &= -7 \\ t_{33} &= 18 + (33-1)(-7) \\ &= 18 + (33)(-7) \\ &= 18 - 224 \\ &\boxed{t_{33} = -206} \end{aligned}$$

$$t_n = 18 + (n-1)(-7)$$

Example:

Ex.3: Find the # of terms in the sequence.

$$\underbrace{31,\ 27}_{-4},\underbrace{27,\ 23}_{-4},\underbrace{23,\ 19}_{-4},\ldots,-53$$

$\therefore$ arithmetic

1. Make a general term.
2. Substitute in -53.
3. Solve for n.

$$a = 31, \quad d = -4$$
$$t_n = 31 + (n-1)(-4)$$
(sub in -53 for t_n)
$$-53 = 31 + (n-1)(-4)$$
(solve for n)
$$84 = (n-1)(-4)$$
$$21 = n - 1$$
$$22 = n$$
$\therefore$ There are 22 terms

Example:

Ex.4 For an arithmetic sequence, $t_7 = 53$ and $t_n = 97$. Find t_{100}.

Solution (1) :

$$t_7 = 53$$
$$a + (n-1)d = 53$$
$$a + (7-1)d = 53$$
$$a + 6d = 53 \quad (1)$$

$$t_{11} = 97$$
$$a + (11-1)d = 97$$
$$a + 10d = 97 \quad (2)$$

Using elimination

$$(2) \quad a + 10d = 97$$
$$(1) \quad a + 6d = 53$$

$$4d = 44$$
$$d = 11$$

Substitute $d = 11$ into (1)

$$a + 6(11) = 53$$
$$a = 53 - 66$$
$$a = -13$$

$$t_n = -13 + (n-1)(11)$$
$$t_{100} = -13 + (99)(11)$$
$$t_{100} = 1076$$

Once we found $d = 11$,

$$t_7 + 93d = t_{100}$$
$$53 + 93(11) = t_{100}$$
$$t_{100} = 1076$$

Or,

$$t_{100} = t_{11} + 89d$$
$$= 97 + 89(11)$$
$$= 1076$$

Solution ② :

$$\underbrace{t_{11} - t_7 = 4d}_{\text{proof}}$$

$$(a + 10d)(a + 6d)$$
$$= 10d - 6d$$
$$= 4d$$

$$t_{11} - t_7 = 4d$$
$$97 - 53 = 4d$$
$$44 = 4d$$
$$11 = d$$

$$t_{100} = t_7 + 93d$$
$$= 53 + 93(11)$$
$$= 1076$$

Example:

Ex.5: For an arithmetic sequence: $t_4 = 19$ and $t_{21} = -49$. Find $t_{38:}$

$$t_{21} - t_4 = 17d$$
$$-49 - 19 = 17d$$
$$-68 = 17d$$
$$-4 = d$$

$$t_{38} = t_{21} + 17d$$
$$= 49 + 17(-4)$$
$$t_{38} = -117$$

$$t_{38} = t_4 + 34d$$
$$= 19 + 34(-4)$$
$$= -117$$

Arithmetic sequence (cont'd)

Recursive sequence $\Rightarrow$ a sequence for which one term(or more) is given and each successive term is determined from the previous term(s).

For an arithmetic formula sequence the recursive formula:

$$\boxed{t_1 = a, t_n = t_{n-1} + d, n \in \mathbb{N}, n > 1}$$

Recall: 1, 1, 2, 3, 5, 8

$$t_1 =, t_2 = 1, t_n = t_{n-1} + t_{n+2}, n \in \mathbb{N}, n > 2$$

Example:

Ex.1: Write the recursive formula for each arithmetic sequence

a. 5,11,17,23, ...

$$t_1 = 5, t_n = t_n = t_{n-1} + 6, n \in \mathbb{N}, n > 1$$

b. $\frac{1}{2}$, $\frac{-1}{2}$, $\frac{-3}{2}$, $\frac{-5}{2}$, ...

$$t_1 = \frac{1}{2}, t_n = t_n = t_{n-1} + 1, n \in \mathbb{N}, n > 1$$

Example:

Ex.2: Write the first 4 terms of the sequence

a. $t_1 = 80, t_n = t_{n-1} - 8, n \in \mathbb{N}, n > 1$

$$80, 72, 64, 56, \ldots$$

7.2 Geometric sequence

Geometric sequence

⇒ Recursive sequence in which noew terms are created by multiplying the previous terms by the same value(common ratio).

Example:

2, 6 , 18, 54, . . .

a=2 first term   Common ratio (r)=3 since $\frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = 3$

General term ⇒ $t_n = ar^{n-1}$

$\therefore t_1 = ar^{1-1}$	$t_2 = ar^{2-1}$	$t_3 = ar^{3-1}$	$t_4 = ar^{4-1}$	AND
$t_1 = ar^0$	$t_2 = ar^1$	$t_3 = ar^2$	$t_4 = ar^3$	SO
$t_1 = a$	$t_2 = ar$			ON!

$$\overbrace{a , ar}^{\times r}, \overbrace{ar , ar^2}^{\times r}, \overbrace{ar^2 , ar^3}^{\times r}, \overbrace{ar^3 , ar^4}^{\times r}, \ldots$$

Recursive Formula:

$$t_1 =, t_n = (t_{n-1})(r), n \in \mathbb{N}, n > 1$$

Example:

Ex.1: For the sequence: $2, 6, 18, 53$

(a) Determine recursive formula:

$$t_1 = 2, \quad t_n = (t_{n-1})(3), n \in \mathbb{N}, \ n > 1$$

(b) General term:

$$t_n = 2(3)^{n-1}$$

(c) t_{10}

$$\begin{aligned} t_{10} &= 2 \cdot 3^{10-1} \\ &= 2 \cdot 3^9 \\ &= 39,366 \end{aligned}$$

7.3 Arithmetic Series

Understanding Arithmetic Series

In the enchanting realm of numbers, where mathematical secrets unfold, arithmetic series takes center stage—a symphony of terms in an arithmetic dance with a prescribed number of steps.

The Tale of Mr. L. Lenarduzzi

Allow me to transport you to the nostalgic corridors of my school days, where the protagonist is none other than Mr. L. Lenarduzzi, our revered math maestro. The tale unfurls during my 11 grade adventures when Mr. Lenarduzzi unraveled a mesmerizing story about the arcane wonders of arithmetic series.

"In the 3rd grade, amidst the realms of multiplication and division within 100, my teacher, whom I'll refer to as 'my teacher,' presented me with a worksheet—a challenge I met with swift prowess. 'I finished it, ma'am,' I declared confidently.

Surprising her, she handed me another, and then another, as my quick triumphs seemed to amuse and intrigue. To test my mettle further, she laid down a gauntlet: write down the numbers from 1 to 100 and find their sum. Unfazed, I embraced the challenge, boldly declaring the sum as 5050.

Intrigued and desiring to unveil the mystery of my method, she beckoned me to the board. There, I began illustrating the series:"

$$1 + 2 + 3 + \ldots + 98 + 99 + 100$$
$$100 + 99 + 98 + \ldots + 3 + 2 + 1$$

$$101 + 101 + 101 + \ldots + 101 + 101 + 101$$

With the class held in suspense, the teacher inquired, 'Where's the answer?' I calmly responded, 'I'm not done yet,' and she patiently agreed, saying, 'Okay, I'm waiting.' Continuing, I unveiled the calculation.

In the enchanting realm of mathematics, Mr. L. Lenarduzzi, a maestro of numbers, found himself at the intersection of curiosity and calculation. Eager to unravel the mysteries of arithmetic, he embarked on a journey to summon the elusive formula for the sum of the first 100 natural numbers:

$$\frac{n \times (n+1)}{2}$$

. The cryptic allure of this formula lay in its ability to unveil the cumulative magic hidden within a sequence, where n represented the final term.

With a twinkle of mathematical intuition, Mr. Lenarduzzi delved into the heart of the formula, substituting $n = 100$ and conjuring forth the mystical expression:

$$\frac{101 \times 100}{2}$$

As the room fell silent, Mr. Lenarduzzi initiated his mathematical incantation:

$$\frac{101 \times \cancelto{50}{100}}{\cancel{2}}$$

A wave of understanding cascaded through the students. What was this magical transformation? Mr. Lenarduzzi, wearing an enigmatic smile, unraveled the mystery.

"Behold the power of cancellation," he proclaimed. "See how the common factor of 2 gracefully cancels out, leaving us with a simplified expression."

The chalkboard now showcased the enchanting result:

$$\boxed{5050}$$

A curious student queried, 'But what happened to the 100?'

"With a touch of mathematical finesse," Mr. Lenarduzzi explained, "we transformed the 100 into its essence, revealing its secret identity as 50. By canceling out the common factor, we unveiled the faster path to our answer – the mystical number 5050."

The students marveled at the elegance of this mathematical metamorphosis. The chalkboard, now not just a canvas for numbers but a portal to a world of mathematical wonders, held a narrative of discovery and revelation.

And so, in the echoes of that enchanted classroom, the legend of cancellation lived on – a tale whispered among students as a key to unlocking the wonders hidden within arithmetic.

As whispers of awe spread among the students, a glimmer of uncertainty crossed 'my teacher's' face. She had handed a prodigious mind a challenge, and now, she grappled with the realization that her student's mathematical prowess surpassed even her expectations. Acknowledging his genius, she adorned him with a medal, a silent tribute to the prodigy who had illuminated the classroom with the brilliance of arithmetic.

Arithmetic Series

An arithmetic series is the sum of the terms in an arithmetic sequence with a definite number of terms.

Formula:

$$S_n = \frac{\overset{\text{\# of terms}}{\overset{\uparrow}{n}}\ (\overset{\text{first}}{\overset{\uparrow}{t_1}} + \overset{\text{last}}{\overset{\uparrow}{t_n}})}{2}$$

- S_n is partial.
- Sum the first
- n terms of a sequence.

↳ Series → The sum of the terms of a sequences.

For Ex: The sequence $2, 4, 8, \ldots$ is an arithmetic sequence, and its sum $2 + 4 + 8 + \ldots$ forms an arithmetic series.

Example:

Ex.1: Find the sum: 10+20+30+...+150

Sol'n:

$$S_n = \frac{15(10+150)}{2}$$
$$S_n = \frac{15(160)}{2}$$
$$= 15(80)$$
$$= 1200$$

$$S_n = \frac{n(t_1 + t_2)}{2}$$

Works well when know the first, last and # of terms. Replace: t_1 with "a" t_n with "a+(n-1)d".

$$S_n = \frac{n[a + a(n-1)d]}{2} \quad \Rightarrow \quad S_n = \frac{n[2a + (n-1)d]}{2}$$

Property of Arithmetic Sequences

$$t_a + t_b = t_c + t_d \quad \text{if } a + b = c + d$$

For Ex:
$t_4 + t_{10} = t_6 + t_8$

$$\begin{aligned} t_4 &= a + 3d \\ t_{10} &= a + 9d \\ \hline t_4 + t_{10} &= 2a + 12d \end{aligned} \qquad \begin{aligned} t_6 &= a + 5d \\ t_8 &= a + 7d \\ \hline t_6 + t_8 &= 2a + 12d \end{aligned}$$

Property for Geometric Sequence

$$t_a \cdot t_b = t_c \cdot t_d \quad \text{if} \quad a + b = c + d$$

$$\left.\begin{aligned} t_5 &= ar^4 \\ t_8 &= ar^7 \end{aligned}\right\} t_5 \cdot t_8 = ar^4 \cdot ar^7 = a^2 r^{11}$$

For Ex:

$$t_5 \cdot t_8 = t_8 \cdot t_{11}$$

Or

$$t_{10} \cdot t_{15} = t_5 \cdot t_{20}$$

$$\left.\begin{aligned} t_2 &= ar \\ t_8 &= ar^{10} \end{aligned}\right\} t_2 \cdot t_{11} = ar \cdot ar^{10} = a^2 r^{11}$$

7.4 Geometric Series

Geo Sequence:

$$2, 6, 18, 54, 162, \ldots$$

with $r = 3$

$$-\frac{1}{3}, 2, -16, 128, \ldots$$

with $r = -8$

Geo Series:

$$2 + 6 + 18 + 54 + 162 + \ldots$$

$$-\frac{1}{4} + 2 - 16 + 128 - \ldots$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Resources

- Grade 11 Review - Jensen Math
- PrepAnywhere
- Functions Notes By Handwriting
- Nelson Functions 11
- Harcourt Mathematics 11 - Functions Relations